Disc Math - §14 - Permutations and Combinations. The Binomial Theorem.

rem: 4 digit PIN's with no repeated digits.

$$10\cdot 9\cdot 8\cdot 7 = {}_{10}P_4 = \frac{10!}{(10-4)!}$$

Def: The number of permutations of r distinct objects chosen from n distinct objects is

$$_nP_r = \frac{n!}{(n-r)!} \quad \text{(permutations)}$$

Ex: How many permutations of n distinct objects?

$$_nP_n = \frac{n!}{(n-n)!} = n!$$

Ex: Jenny manages 20 programmers at NowP Inc. In how many ways can she select 8 programmers from the 20 for different programming jobs?

$$_{20}P_8 = \frac{20!}{12!} = 5.079 \times 10^9.$$

Def: The number of ways to select r objects from a set of n objects (order doesn't matter) is

$$_nC_r = \frac{n!}{(n-r)!\,r!} \quad \text{(combinations).}$$

Ex: Jane manages 20 programmers at NowQ Inc. In how many ways can she select 8 programmers to work as a team

$$_{20}C_8 = \frac{20!}{12!\,8!} = 125970$$

Ex: Compute $_{100}C_2$, $_{120}C_2$, $_{300}C_{299}$.

Ex: a) How many poker hands b) How many full house poker hands

a) $_{52}C_5 = \frac{52!}{47!\,5!} = 2598960$ b) $_{13}C_1\ {}_4C_3\ {}_{12}C_1\ {}_4C_2 = 3744$

$p = 3744/2598960 = 0.001441$

Disc Math - §14 - Permutations and Combinations

Binomial Theorem

Ex: How many 6/49 selections? What is the probability of winning the jackpot on one ticket.

$$_{49}C_6 = \frac{49!}{43!\,6!} = 13983816$$

Ex: A shipment of 35 PC's contains 6 defective. 28 PC's are selected for installation. What is the probability that the selected PC's will contain 4 defective PC's?

$$_{35}C_{28} = 6724520$$

$$_6C_4 \cdot {}_{29}C_{24} = 1781325 \qquad p = \frac{1781325}{6724520} = 0.265$$

Ex: A committee of 5 students is to be selected from a group containing 9 freshmen and 14 sophomores.

a) How many committees with at most 1 freshman are possible

b) How many committees with at least 1 freshman are possible

Sol: a) $_9C_1 \cdot {}_{14}C_4 + {}_{14}C_5 = 9009 + 2002 = 11011$

b) $_{23}C_5 - {}_{14}C_5 = 33649 - 2002 = 31647.$

Def: Suppose that there are n objects, of which n_1 are indistinguishable, ..., n_k are indistinguishable with $n_1+n_2+\dots+n_k=n$. The number of distinct permutations of these n objects is

$$\frac{n!}{n_1!\,n_2!\,\dots\,n_k!}$$

Ex: How many strings of length 7 with the symbols e, e, e, b, b, c, d

$$\frac{7!}{3!\,2!\,1!\,1!} = 420$$

Disc Math · §14 · Permutations and Combinations. Binomial Theorem.

Premutations and Combinations with Replacement.

rem: The number of permutations of r objects out of n distinct objects with repetitions allowed is n^r.

Ex: A jeweler designing a necklace has decided to use five stones from a supply of diamonds, rubies and emeralds. How many sets of stones are possible?

Sol: No order → combinations problem: combinations of 5 objects out of 3 objects with repetitions allowed.

2 diamonds, 2 rubies, 1 emerald → $**|**|*$

5 diamonds. → $*****||$

Thus we are counting ways to choose 5 items out of 7 → ${}_7C_5$

Def: To represent a combination of r objects out of n distinct objec with repetition allowed, we need $n-1$ makers →

$$ {}_{r+n-1}C_r = \frac{(r+n-1)!}{r!\,(n-1)!} $$

Ex: 7 programers set one task each: C++, Java or Perl. How many sets of assignments are possible.

$$ {}_9C_7 = \frac{9!}{7!\,2!} = 36. $$

Binomial Theorem.

rem: $(a+b)^2 = a^2+2ab+b^2$; $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$; $(a+b)^n$:

$(a+b)^n = (a+b)(a+b)\dots(a+b) = {}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1} b^1 + \dots + {}_nC_{n-1}\, a^1 b^{n-1} + {}_nC_n\, a^0 b^n$

Th: (Binomial Theorem) $\boxed{(a+b)^n = \sum_{k=0}^{n} {}_nC_k\, a^{n-k} b^k}$

Disc Math - §14 - Permutations & Combinations

Binomial Theorem.

rem: ${}_nC_r$ are also known as binomial coefficients.

ex: $(x-2)^5 = {}_5C_0\, x^5(-2)^0 + {}_5C_1\, x^4(-2)^1 + {}_5C_2\, x^3(-2)^2 + {}_5C_3\, x^2(-2)^3$

$+ {}_5C_4\, x^1(-2)^4 + {}_5C_5\, x^0(-2)^5 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

Cor: $(1+1)^n = {}_nC_0 + {}_nC_1 + \dots + {}_nC_{n-1} + {}_nC_n$

$\Rightarrow \sum_{i=0}^{n} {}_nC_i = 2^n$

Notice ${}_nC_i$ is the number of i-element subsets of a n-element set. The total number of subsets of an n-element set is 2^n!

Th: (Pascal's formula).

$$\boxed{{}_nC_k = {}_{n-1}C_{k-1} + {}_{n-1}C_k}$$

Pr: ${}_nC_k$ - select k objects out of n. Split along one of two possibilities: item 1 is one of the k selected $\rightarrow {}_{n-1}C_{k-1}$ ways to selected the remaining $k-1$; item 1 is not one of the selected $\rightarrow {}_{n-1}C_k$ choices. ☒

Cor: Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
- & & - & & - & & - & & - & & - & & -
\end{array}
$$

${}_nC_k$

HW

§4.4 p 288

12, 26, 34, 40, 52, 66

74, 76, 82, 92, 96

§4.5 p 299

2, 14, 17, 21