Disc Math - §16 - Functions

Function is just a special binary relation from a set S to a set T.

Def: Let S and T be sets. A function $f: S \to T$ is a subset of $S \times T$ where each element of S appears exactly once as the first component of an ordered pair. S is the domain and T is the codomain of the function. The set of all elements of the codomain which appear as second component of an ordered pair is the range of the function.

ex: $f: \mathbb{Z} \to \mathbb{Z}$, $f(z) = z^2 + 1$ has a range $\mathbb{N}$. Some of the pairs are:

$$\dots (-2,5), (-1,2), (0,1), (1,2), (2,5), \dots$$

ex: $f: \mathbb{Z} \to \mathbb{Z}$, $f(z) = \sqrt{z}$ is not a function, but

$f: \mathbb{N} \to \mathbb{R}$, $f(x) = \sqrt{x}$ is.

The relation $x \sim y$ iff $y^2 = x$ does not define a function on $\mathbb{R}$.

rem: Of course there are also multivariate functions:

e.g. $f: \mathbb{N} \times \mathbb{Z} \to \mathbb{Q}$, $f(x,y) = x^y$ is a function of 2 variables

ex: The modulo function $f(x,n) = x \bmod n$ gives value r, where $x = qn + r$, $0 \le r < n$.

e.g. $25 \bmod 7 = 4$; $-17 \bmod 5 = 3$; $1024 \bmod 8 = 0$.

Def: A function $f: S \to T$ is onto (surjective) if the range of f equals the codomain of f.

ex: $f: \mathbb{Z} \to \mathbb{Z}$, $f(z) = z^2 + 1$ is not onto.

$f: \mathbb{Z} \to \mathbb{N}$, $f(z) = z^2 + 1$ is onto.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ is not onto.

Def: A function $f: S \to T$ is one-to-one (injective), if no element of T is the image under f of two distinct elements of S.

ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ is one-to-one.

ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is not one-to-one.

Def: A function $f: S \to T$ is a bijection if it is both one-to-one and onto.

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a bijection.
$f: \mathbb{R} \to [-1, 1]$, $f(x) = \sin x$ is not a bijection.

Def: Let $f: S \to T$ and $g: T \to U$ be two functions. Then the composite funct $g \circ f$ is a function $S \to U$ defined by $(g \circ f)(s) = g(f(s))$.

Ex: $f, g: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$, $g(x) = -x^2$, $f \circ g\,(x) = e^{-x^2}$, $g \circ f(x) = -e^{2x}$

Th: The composition of two bijections is a bijection.

Pr: Trivial / Book. ☒

Def: Let $f: S \to T$ be a function. If there exists a function $g: T \to S$ s.t. $g \circ f = i_S$ and $f \circ g = i_T$, then g is called the inverse function of f and is denoted by f^{-1}.

Ex: $f: \mathbb{R} \to \mathbb{R}_+$, $f(x) = e^x$; $f^{-1}: \mathbb{R}_+ \to \mathbb{R}$, $f^{-1}(x) = \ln x$.

Permutations.

Def: A bijection from a set A onto itself is called a permutation.

not: A permutation of the set $A = \{1, 2, 3, 4\}$ could be represented as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \quad f(1) = 3 \text{ etc.}$$

Permutations are also written in cycle notation $(1, 3, 4, 2)$; each element is mapped onto the one to the right and the last one onto the first. (Fixed elements are not linsted).

Ex: On $A = (1, 2, 3, 4, 5)$ the permutation $(1\ 4)(3\ 5)$ stands for

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

rem: The composition of two permutations of the same set is again a permutation of the same set.

Ex: $A = \{1, 2, 3, 4\}$, $f = (1, 2, 3)$, $g = (2, 3)$. $g \circ f = (2, 3) \circ (1, 2, 3) = (1, 3)$

(see back)

Def: A permutation on a set that maps no element to itself is called a derangement.

Ex: $A = \{1,2,3,4,5\}$, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$ is a derangement.

How many functions?

rem: If S, T are finite sets with $|S| = m$, $|T| = n$ the the number of relations from S to T is 2^{mn}.

Th: If $|S| = m$ and $|T| = n$, then

1) The number of functions $f: S \to T$ is n^m.
2) Assuming that $m \le n$ the number of one-to-one functions $f: S \to T$ i
$$\frac{n!}{(n-m)!} = {}_nP_m$$
3) Assuming that $m \ge n$ the number of onto functions $f: S \to T$ is
$$n^m - {}_nC_1(n-1)^m + {}_nC_2(n-2)^m - {}_nC_3(n-3)^m + \dots + (-1)^{n-1}{}_nC_{n-1}(1)^m.$$
4) Assuming that $m = n$ the number of bijections $f: S \to T$ is $m!$

Pr: 1) For each of the m elements of S we have n choices for an image
$$\underbrace{n \times n \times \dots \times n}_{m} = n^m.$$

2) As in 1) but now the choices are $n(n-1)(n-2)\dots(n-m+1) = {}_nP_m$

3) We substract the number of non-onto functions from the total number of functions. Enumerate the elements of the set T as $t_1, t_2, \dots, t_n$. For i, $1 \le i \le n$ let A_i denote the set of functions $S \to T$ that do not map anything to t_i. By the principle of inclusion/exclusion we have
$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| -$$
$$\dots + (-1)^{n-1} |A_1 \cap A_2 \cap \dots \cap A_n|. \quad (*)$$

From (1), $|A_i| = (n-1)^m$ and in the first term of (*) there are ${}_nC_1 = n$ such terms. Similarly $|A_i \cap A_j| = (n-2)^m$ and there are ${}_nC_2$ such term in (*). A similar result holds for all intersection terms. Hence,

$|A_1 \cup A_2 \cup \ldots \cup A_n| = {}_nC_1 (n-1)^m - {}_nC_2 (n-2)^m + {}_nC_3 (n-3)^m - \cdots + (-1)^{n-1} {}_nC_n (n-n$

This is the number of functions that fail to map to at least of the elemen of T. Thus the number of onto functions is $n^m - |A_1 \cup A_2 \cup \cdots \cup A_n|$ which establishes (3).

(4) follows immediately from (2) when $n = m$. ☒

Ex: $S = \{a, b, c\}$, $T = \{a, b\}$.

1) The number of injections $T \to S$ is ${}_3P_2 = \frac{3!}{1!} = 6$

2) The number of surjections $S \to T$ is

$$2^3 - {}_2C_1 (1)^3 = 8 - 2 = 6$$

3) The number of functions $S \to T$ is $2^3 = 8$.

4) The number of functions $T \to S$ is $3^2 = 9$.

HW §5.4. p 403

18, 24, 40, 49, 57, 64, 67, 74, 75