Disc Math - §18 - Graphs

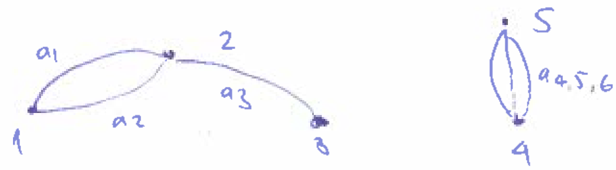
Def: A graph is an ordered triple (N, A, g) where,

N - is the (nonempty) set of nodes (vertices)

A - is the set of arcs (edges)

$g: A \to$ unordered pairs of nodes; $g(a)$ are the endpoints of the arc a.

ex:



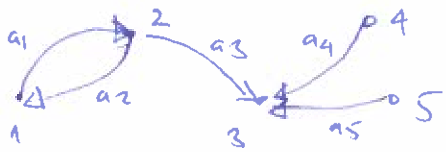
Def: A directed graph (digraph) is an ordered triple (N, A, g) where,

N - nonempty set of nodes

A - set of arcs

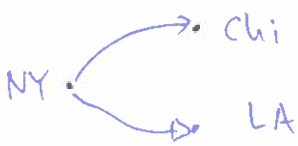
$g: A \to N \times N$, $g(a) = (x, y)$, x - initial, y - terminal point of a.

ex:

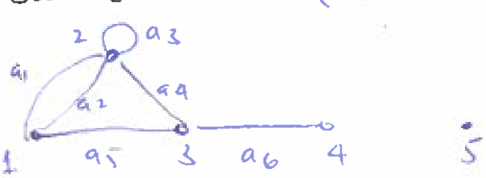


$g(a_5) = (5, 3)$.

def extensions: Labelled graph has labels on the nodes; Weighted graph has numerical values on the arcs.



Def: Two nodes in a graph are adjacent if they are endpoints of the same arc. (1 and 3 are adjacent, but 1 and 4 are not).



A loop is an arc for which the two endpoints coincide (a_3 is a loop).

A graph with no loops is loop-free.

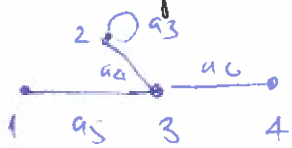
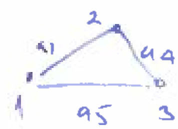
Two arcs with the same endpoints are parallel arcs (a_1, a_2 above).

A simple graph is one with no loops or parallel arcs.

An **isolated node** is a node adjacent to no other node (5).

The **degree** of a node is the number of arcs that end in that node:
(1 has degree 3; 2 has degree 5; 5 has degree 0).

A **subgraph** consists of a subset of the nodes and a subset of the arcs with the same endpoints assignment as in the original graph.

(e.g.  or ).

Def: A **path** from a node n_0 to a node n_k is a sequence

$$n_0, a_0, n_1, a_1, \ldots, n_{k-1}, a_{k-1}, n_k$$

of nodes and arcs s.t. $g(a_i) = (n_i, n_{i+1})$. The **length** of a path is the number of arcs it contains.

ex: $(2)\, a_1\, (1)\, a_5\, (3)\, a_4\, (2)\, a_3\, (2)\, a_4\, (3)\, a_6\, (4)$ is a path of length 6.

A graph is **connected** if there is a path from any node to any other node

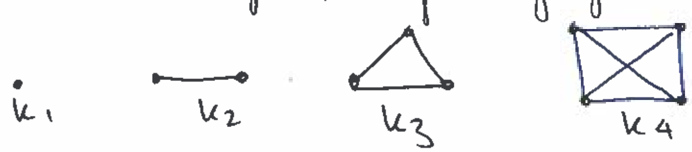
Def: A **cycle** is a closed path from a node n_0 back to n_0, where no arc appears more than once, n_0 occurs only at the ends and no other node appears more than once.

ex: $(1)\, a_1\, (2)\, a_4\, 3\, (a_5)\, 1$ is a cycle; so is $(2)\, a_3\, (2)$.

A graph with no cycles is called **acyclic**.

def: A **complete graph** is one in which any two distinct nodes are adjacent.

ex: Here are the simple, complete graphs on $n = 1, 2, 3, 4$ vertices.

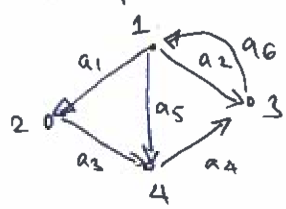


? (stud).
K_5

Def: A simple graph is a **bipartite complete graph** if its nodes can be partitioned into two disjoint nonempty sets N_1 and N_2 s.t. two nodes x and y are adjacent iff $x \in N_1$ and $y \in N_2$. If $|N_1| = m$ and $|N_2| = n$, such a graph is denoted by $K_{m,n}$.

Ex:  $K_{4,2}$. Stud. draw $K_{3,3}$.

Def: In a directed graph, a path from node n_0 to node n_k is a sequence
$n_0, a_0, n_1, a_1, \dots, n_{k-1}, a_{k-1}, n_k$
where n_i is the initial point and n_{i+1} is the terminal point of a_i. Cycle is a closed path with no repetitions except the endpoints.

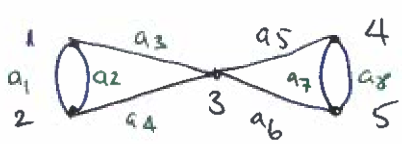
Ex: 

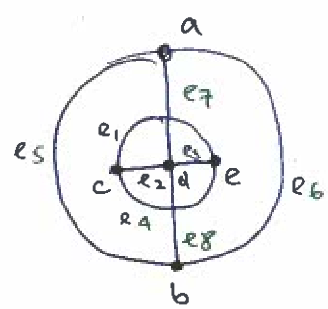
$(1) a_1 (2) a_3 (4) a_4 (3) a_6 (1)$ is a cycle.
Is (2) reachable from (3)? Yes, along the path $(3) a_6 (1) a_1 (2)$.

Ex: These two graphs are presented differently, but they are not different:

Def: Two graphs (N_1, A_1, g_1) and (N_2, A_2, g_2) are isomorphic if there are bijections $f_1 : N_1 \to N_2$ and $f_2 : A_1 \to A_2$ s.t. $\forall a \in A_1$, $g_1(a) = (x, y)$ iff $g_2[f_2(a)] = (f_1(x), f_1(y))$.

Ex: Find the isomorphism:





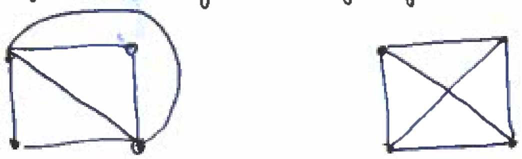
$f_1 : 1 \to c$; $2 \to e$; $3 \to d$; $4 \to b$; $5 \to a$

$f_2 : a_1 \to e_1$, $a_2 \to e_4$, $a_3 \to e_2$, $a_4 \to e_3$, $a_5 \to e_8$, $a_6 \to e_7$, $a_7 \to e_5$, $a_8 \to e_6$.

Th: (Isomorphism for simple graphs). Two simple graphs (N_1, A_1, g_1) and (N_2, A_2, g_2) are isomorphic if there is a bijection $f : N_1 \to N_2$ s.t. $\forall n_1, n_2 \in N_1$, n_1 and n_2 are adjecent iff $f(n_1)$ and $f(n_2)$ are adjacent.

Pr: There is at most one arc between any pair of nodes. ☒

Ex: The following two graphs are not isomorphic.

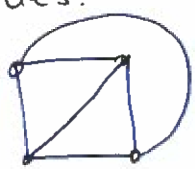


(The 1st has parallel arcs and the second one doesn't).

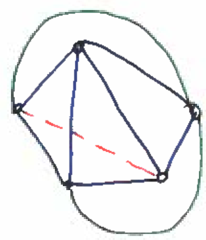
Planar Graphs.

Def: A graph is planar if it can be drawn on a plane so that arcs intersect only at nodes.

Ex: K_4 is planar:



Ex: K_5 is not planar:

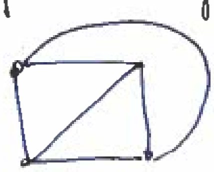


(Maybe it can be done in some other way. We need a rigorous proof!).

Th: (Euler) For a (simple) connected planar graph with n nodes and a arcs, if the planar representation divides the plane into r regions, the

$$n - a + r = 2. \quad \text{(Euler's formula)}.$$

Ex: Before presenting the proof let's look at an example.



$n = 4, \ a = 6, \ r = 4$

$4 - 6 + 4 = 2.$

Pr: By Induction on the number of arcs a.

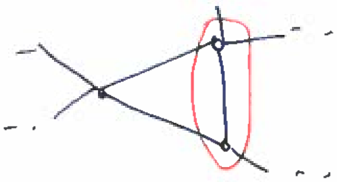
Base case $a = 0$. Then $n = 1$, $r = 1$ ✓

Assume the formula holds for the planar representation of any (simple) connected planar graph with k arcs. Consider such a graph with $k+1$ arcs.

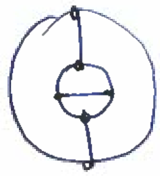
Case 1: The $k+1$ arc graph has a node of degree 1:

Erasing this node leaves a graph with k arcs for which $n - a + r = 2$.

For the $k+1$ arc graph $(n+1) - (k+1) + r = 2$ is then true.

Case 2: The $k+1$ arc graph has no node of degree 1. The we erase an are like this:  From $n-a+r=2$ we have (Inductive Hyp.)

$n-(a+1)+(r+1)=2$. ☒

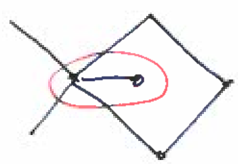
ex:  $n=6,\ a=9,\ r=5$

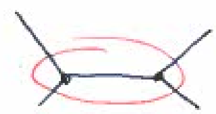
$6-9+5=2$.

Cor: For a simple, connected, planar graph with $n \geq 3$ nodes and a arcs:

i) $a \leq 3n-6$

ii) If there are no cycles of length 3, then $a \leq 2n-4$.

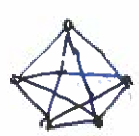
Pr: i) In a planar representation of such a graph we will count the number of boundary arcs to each region. Arcs which are wholly interior to a region  contribute twice as boundary arcs for that region

Arcs which separate two regions  also contribute twice (once for each region). Therefore the number of boundary arcs is $2a$.

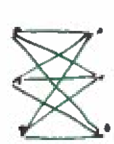
There are no regions with exactly one boundary arc since there are no loops. There are no regions with exactly two boundary arcs since there are no parallel arcs. Thus each region has at least 3 boundary arcs, so $3r$ is the minimum number of boundary arcs $\Rightarrow 2a \geq 3r$.

$$2a \geq 3(2-n+a) = 6-3n+3a \Rightarrow a \leq 3n-6$$

ii) If there are no cycles of length 3, then each region has at least 4 adjacent boundary edges so $2a \geq 4r \Rightarrow a \leq 2n-4$. ☒

ex: K_5 is not planar:  $n=5,\ a=10$

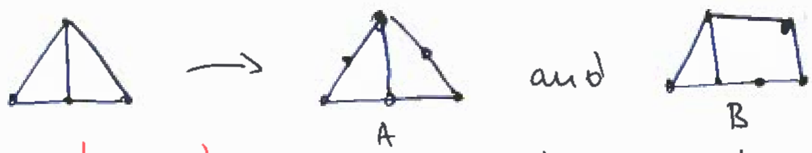
$a \overset{?}{\leq} 3n-6$, $\quad 10 \overset{?}{\leq} 3(5)-6=9$. False.

ex: $K_{3,3}$ is not planar:  $n=6,\ a=9$. No cycles of length 3.

$a \overset{?}{\leq} 2n-4$, $\quad 9 \leq 2\cdot 6-4=8$. False.

Def: Two graphs are homeomorphic, if they both can be obtained from the same graph by a sequence of elementary subdivisions, in which a single arc $x-y$ is replaced by two new arcs $x-z$ and $z-y$ where z is a new node.

Ex:



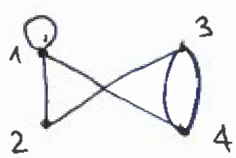
Th: (Kuratowski) A graph is nonplanar if it contains a subgraph that is homeomorphic to K_5 or $K_{3,3}$. ☒

Computer representations of graphs → Adjacency Matrix.

Def: Consider a graph with k nodes ordered as $1, 2, \ldots, k$. The adjacency matrix of the graph wrt this ordering has

$a_{ij} = p$, where p is the number of arcs between nodes i and j. (symmetric)

Ex:



$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

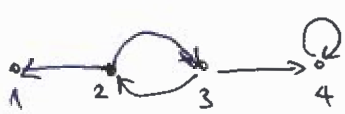
Ex: (Stud). Write the adjacency matrix of K_4:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Def: For a directed graph the adjacency matrix is:

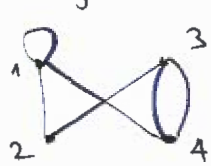
$a_{ij} = p$, where p is the number of arcs from node i to node j.

Ex:



$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

rem: large sparse adjacency matrices can be compactified to adjacency lists as in:



$1 \to \{1, 2, 4\}$ $\quad 2 \to \{1, 3\}$

$3 \to \{2, 4, 4\}$ $\quad 4 \to \{1, 3, 3\}$.

HW §6.1 p 499

8, 14, 17, 26, 28, 30, 34, 38, 42

46, 52, 66, 80