Disc Math - §2 - Propositional Logic (Dynamics).

Def: An argument is a sequence of statements that are called premises, except the final statement which called a conclusion. Notation:

$$p_1, p_2, \dots, p_n \vdash q \quad \text{or} \quad p_1 \wedge p_2 \wedge \dots \wedge p_n \rightarrow q.$$

Q?: When is an argument valid?

Def: An argument is valid if when all the premises are true, then the conclusion is also true.

Ex: Is this a valid argument? $p \rightarrow q \vee \sim r,\ q \rightarrow p \wedge r \vdash p \rightarrow r$

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \rightarrow q \vee \sim r$	$q \rightarrow p \wedge r$	$p \rightarrow r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	T	F
T	F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Invalid Argument.

Th: An argument $p_1, p_2, \dots, p_n \vdash q$ is valid iff $p_1 \wedge p_2 \wedge \dots \wedge p_n \rightarrow q$ is a tautology.

Def: An argument consisting of two premises and a conclusion is called a syllogism: $p, q \vdash r$.

Def: (Modus Ponens) $p \rightarrow q,\ p \vdash q$

$((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Valid!

Ex: If n is a prime then n is integer, 19 is prime $\vdash$ 19 is an integer.

If $\sqrt{2}$ is prime then $\sqrt{2}$ is integer, $\sqrt{2}$ is prime $\vdash$ $\sqrt{2}$ is integer.

Both these arguments are valid, but the second is not sound.

Def: An argument is called sound if it is valid and its premises are true. Otherwise an argument is unsound.

Def: (Modus Tollens) $p \to q, \sim q \vdash \sim p$

p	q	$\sim q$	$p \to q$	$\sim p$
T	T	F	T	F
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Valid.

ex: If 24 is divisible by 10, Then it is divisible by 5. 24 is not divisible by 5. Therefore 24 is not divisible by 10.

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rem: There are many examples of invalid arguments. We look at two.

Def: (Converse error) $p \to q, q \vdash p$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Invalid.

ex: If Maya codes HTML, Then Maya has a WebPage. Maya has a WebPage $\vdash$ Therefore Maya codes HTML.

Def: (Inverse Error) $p \to q, \sim p \vdash \sim q$

p	q	$p \to q$	$\sim p$	$\sim q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Invalid.

If 24 is divisible by 10, Then it is divisible by 2. 24 is not divisible by 10. Therefore 24 is not divisible by 2.

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rem: To check the validity of an argument with n elementary statements in the premises we need a truth table with 2^n rows. Instead we can manipulate the premises.

Plus, no truth tables in predicate logic.

We can (and will) use formal logic which uses a system of derivation rules to validate an argument. We want a formal logic system That is correct (only valid arguments are provable) and complete (every valid argument is provable).

The derivation rules are of two types: Equivalence Rules ⊕ Inference Rules.

Equivalence Rules: (substitution is allowed in either direction).

1. $p \vee q \equiv q \vee p$ (1') $p \wedge q \equiv q \wedge p$ commutativity
2. $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (2') $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ associativity.
3. $(p \vee q)' \equiv p' \wedge q'$ (3') $(p \wedge q)' \equiv p' \vee q'$ De Morgan
4. $p \to q \equiv p' \vee q$ implication (5) $(p')' \equiv p$ (6) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

ex: If one of premises is $(p' \vee q') \vee r$ we can rewrite it as follows:

$$(p' \vee q') \vee r \underset{D.M.}{\equiv} (p \wedge q)' \vee r \underset{impl}{\equiv} p \wedge q \to r.$$

Inference Rules: These are unidirectional, they allow us to add to the premises using the inference rules.

① $p, p \to q \vdash q$ Modus Ponens ② $p \to q, \sim q \vdash \sim p$ Modus Tollens

③ $p, q \vdash p \wedge q$ conjunction ④ $p \wedge q \vdash p$ ④' $p \wedge q \vdash q$ simplification

⑤ $p \vdash p \vee q$ ⑤' $q \vdash p \vee q$ addition.

ex: If we have $p \to (q \wedge r)$ and p we can add $q \wedge r$ by Modus Ponens.

ex: Using propositional logic, prove that the argument below is valid:

$$p, (q \to r), p \wedge q \to s \vee r', q \vdash s$$

Here is the proof sequence:

$$\dfrac{\dfrac{q,\ q \to r}{r}\,\text{MP} \qquad \dfrac{\dfrac{\dfrac{p,\ q}{p \wedge q}\,\text{conj} \quad p \wedge q \to s \vee r'}{s \vee r'}\,\text{MP}}{r \to s}\,\text{impl}}{s}\,\text{MP}$$

rem: There might be more then one correct proof sequence.

note: The formal system of derivation rules described so far is both correct and complete. (correctness is obvious!). But many formal systems of propositional logic use additional (truth-preserving) inference rules. We can prove the additional rules using the original rules.

ex: Show that the "Hypothetical syllogism" is a valid argument:

$$p \to q, q \to r \vdash p \to r \quad (p \to q, q \to r, p \vdash r)$$

Here is the proof sequence: $p, p \to q \vdash q$; $q, q \to r \vdash r$. ☒

Def: The deduction method says that if the conclusion is an implication then the validity of $p_1, p_2, .., p_n \vdash q \to r$ is equivalent to the validity of $p_1, p_2, .., p_n, q \vdash r$.

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Verbal arguments:

Ex: Prove that the following argument is valid:

Russia was a superior power, and either France was not strong or Napoleon made an error. Napoleon did not make an error, but if the army did not fail, then France was strong. Hence, the army failed and Russia was a superior power.

$$R,\ F'\vee N,\ N',\ A'\to F \vdash A\wedge R$$

$$\dfrac{F'\vee N \equiv F\to N \qquad N'}{F'}\ MT$$

$$\dfrac{A'\to F \qquad F'}{A}\ MT$$

$$\dfrac{R \qquad A}{A\wedge R}\ \text{conj.}$$

Ex: The Island (of Laval) has two types of people: knights = H who always speak the truth, $H\to T$ and knaves = E who always lie, $E\to L$.

You visit the island and speak to two natives:

Rejean says: Gaetan is a knight

Gaetan says: Rejean and I are of opposite types.

What are they?

Sol: Assume $R\to H$

$$\dfrac{R\to H \qquad H\to T}{R\to T}\ \text{hyp. syll.}$$

$$\dfrac{R\to T}{G\to H \qquad H\to T}\ \text{extra premise}$$

$$\dfrac{G\to H \qquad H\to T}{G\to H \qquad G\to T}$$

$$\dfrac{G\to H \qquad G\to T}{R\to E}$$

Contradiction.

Then we must have $R\to E$.

$$\dfrac{R\to E \qquad E\to L}{R\to L}$$

$$\dfrac{R\to L}{G\to E}\ \text{extra premise.}$$

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\# 15, 19, 22, 25, 27, 31, 35, 38, 44, 52, 54