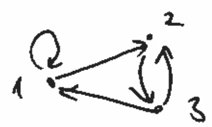
Disc Math - §20 - Directed graphs and binary relations

constr: Let G be a directed graph with n nodes and no parallel arcs. For this kind of graph the adjacency matrix is n×n Boolean matrix i.e. a matrix whose elements are 0's and 1's.

ex: 

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$A_{ij} = 1$ if arc from node i to node j.
0 otherwise.

rem: There is a 1-1 correspondence:

Directed graphs, n nodes, no parallel arcs	$\leftrightarrow$	n×n Boolean matrices

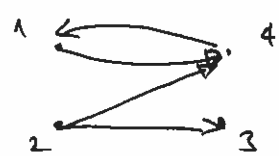
constr: G - directed graph, n nodes, no parallel arcs. Let N be the set of nodes. Define a binary relation on N:

$$n_i \,\rho\, n_j \Leftrightarrow \exists \text{ an arc in } G \text{ from } n_i \text{ to } n_j$$

def: ρ is called the adjacency relation of the graph.

ex: For the graph above $\rho = \{(1,1), (1,2), (2,3), (3,1), (3,2)\}$.

ex: Draw the graph of the binary relation $\{(1,4), (2,3), (2,4), (4,1)\}$ on $N = \{1,2,3,4\}$.



rem: Thus we have 3 bijections:

ρ		G		A
Binary relations on n element sets	$\leftrightarrow$	Directed graphs with n nodes and no parallel arcs	$\leftrightarrow$	n×n Boolean matrices

rem: Properties of the binary relation such as reflexivity, symmetry etc. are reflected in the graph and in the Boolean matrix.

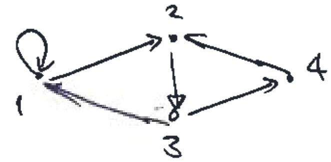
ex: ρ is reflexive, G has a self-loop at every node, A has 1's on the diagonal.

Disc Math - § - Digraphs and binary relations

Goal: We are going to study reachability:

Def: In a directed graph, node n_j is reachable from node n_i, if there is a path from n_i to n_j

Ex: (*)



(1) is reachable from (4).

Digression on the algebra of Boolean matrices:

Def: On $\{0,1\}$ we define

· times

x	y	$x \wedge y$
1	1	1
1	0	0
0	1	0
0	0	0

+ ← plus

x	y	$x \vee y$
1	1	1 ← Not $\mathbb{Z}_2$.
0	1	1
1	0	1
0	0	0

def: On Boolean matrices we have the operations:

$A \wedge B$ (elementwise $\wedge$),

$A \vee B$ (elementwise $\vee$),

$A \times B = C$, $c_{ij} = \bigvee_{k=1}^{m} (a_{ik} \wedge b_{kj})$ ($\times$ to distinguish it from matrix mu

Ex:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

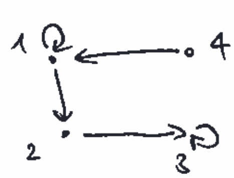
$$A \wedge B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A \vee B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A \times B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

---

rem: Let ρ, σ be binary relations with corresponding matrices R, S.

Then $\rho \cup \sigma \Leftrightarrow R \vee S$; $\rho \cap \sigma \Leftrightarrow R \wedge S$.

Ex: (A)



$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A^{(2)} = A \times A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

→ Write A next to A and explain the composition of paths

Disc Math - 8 - Digraphs and binary relations

(ex) $A^{(3)} = A \times A^{(2)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

length 3 paths.

Th: If A is the Boolean adjacency matrix for a directed graph G with n nodes and no parallel arcs, then $A^{(m)}[i,j] = 1$ iff there is a path of length m from node n_i to node n_j.

Pr: As in the ex. above. ☒

rem: In graph with n nodes, any path with n or more arcs will have at least $n+1$ nodes and therefore repeated nodes and a cycle. For studies of reachability we can ignore the cycles; consequently we never have to look at paths of length more than n.

Def: For a graph G with adjacency matrix A the reachability matrix R is defined by

$$R = A \vee A^{(1)} \vee A^{(2)} \vee \ldots \vee A^{(n)}$$

Cor: In G, n_j is reachable from n_i iff $R[i,j] = 1$.

Ex: (Above, continued).

(ex) $A^{(4)} = A \times A^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

$$R = A \vee A^{(2)} \vee A^{(3)} \vee A^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

rem: If $G \leftrightarrow A \leftrightarrow \rho$ and R is the reachability matrix, the relation corresponding to R, ρ^R is clearly the transitive closure of ρ (if there are composable arcs we add the compositive which is a path of length 2; if there are composable paths, we add the composition).

Disc Math – § – Digraphs and binary relations

(n_i, n_j) belongs to the transitive closure of ρ $\Leftrightarrow$ n_j is reachable from n_i in G $\Leftrightarrow$ $R[i,j]=1$.

omitted

→ transitive closure is the same as establishing the reachability relation

rem: Matrix multiplication is $\Theta(n^3)$, but we need to compute $A^{(2)}, \dots, A^{(n)}$ so $\Theta(n^4)$ Boolean operations. Adding the matrices requires $\Theta(n^3)$ ⇒ the algorithm for computing the transitive closure of a relation (or the reachability matrix of a graph) is $\Theta(n^4)+\Theta(n^3)=\Theta(n^4)$.

end

Warshall's Algorithm.

For a graph with n nodes, Warshall's Algorithm computes a sequence of $n+1$ matrices $M_0, M_1, \dots, M_n$. $\forall k$, $0 \le k \le n$, $M_k[i,j]=1$ iff there is a path in G from n_i to n_j whose interior nodes (not end-nodes) come only from the set of nodes $\{n_1, n_2, \dots, n_k\}$.

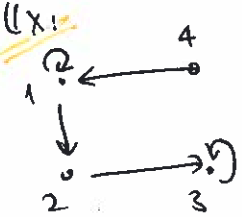
When $k=0$, the paths have no interior nodes → $M_0 = A$.

When $k=n$, the set $\{n_1, n_2, \dots, n_n\}$ has all nodes → $M_n = R$.

Alg: Start with $A = M_0$. The computation proceeds inductively. Assume M_k has been computed; compute M_{k+1}. $M_{k+1}[i,j]=1$ iff there is a path $n_i \to n_j$ whose interior nodes come from the set $\{n_1, \dots, n_{k+1}\}$. This can happen in two ways:

1. $M_k[i,j]=1$; just carry forward 1 entries from M_k into M_{k+1}
2. $M_k[i,k+1] \wedge M[k+1,j]=1$. (This condition can be tested since M_k has been already computed).

Ex:



$$M_0 = A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow r_1 \vee r_1 \\ \Big\} \text{unchanged} \\ \\ \leftarrow r_1 \vee r_4 \end{array}$$

Disc Math - § - Digraphs & Binary Relations

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} \leftarrow r_1 \vee r_2 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{unchanged} \\ \leftarrow r_4 \vee r_2 \end{array} , \quad M_3 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{array}{l} \leftarrow r_1 \vee r_3 \\ \leftarrow r_2 \vee r_3 \\ \leftarrow r_2 \vee r_3 \\ \leftarrow r_4 \vee r_3 \end{array}$$

$$M_4 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \leftarrow \text{unchanged.}$$

rem: This algorithm is $\Theta(n^3)$: going through $M_0, \ldots M_n$ times $2n^2$ operations for updating the entries.

HW §7.1 p 566

4, 6, 14, 18, 28, 34, 38, 42