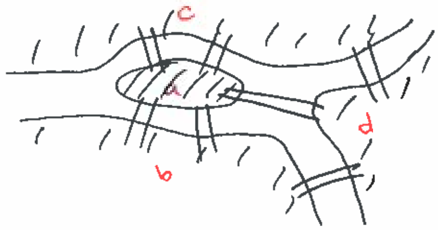
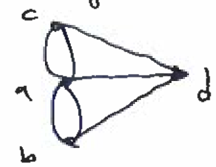
Disc Math - §21 - Euler path and Hamiltonian circuit

Ex: The 7 bridges of Königsberg: could a person walk through the city crossing each bridge only once?



The set up could be represented as a graph:



Def: An Euler path in a graph G is a path that uses each arc of G exactly once. (Nodes can be repeated)

rem: When searching for an Euler path we can assume the graph is connected.

Whether Euler path exists or not depends on the degrees of the nodes.

Th: The number of odd nodes in a graph is even.

Pr: Let $N(k)$ be the number of nodes of degree k. Let S be the sum of the degrees of all nodes in the graph; S - is even

$$S = 1\cdot N(1) + 2\cdot N(2) + \ldots + kN(k) = 1\cdot N(1) + 3\cdot N(3) + 5\cdot N(5) + \ldots + 2\cdot N(2) + 4N(4) + \ldots$$

Thus $1\cdot N(1) + 3\cdot N(3) + 5\cdot N(5) + \ldots = \underbrace{1 + \ldots + 1}_{N(1)} + \underbrace{3 + \ldots + 3}_{N(3)} + \ldots + \underbrace{(2n+1) + \ldots + (2n+1)}_{N(2n+1)}$

must be even. For this sum of odd numbers to be even we must have $N(1) + N(3) + \ldots + N(2n+1) \in \mathbb{Z}$ even. ☒

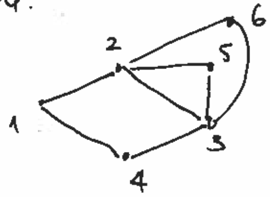
Th: (Euler). An Euler path exists in a connected graph iff either

i) there are no odd nodes or

ii) there are precisely two odd nodes.

For the case of no odd nodes, the path can begin at any node and will end there. For the case of two odd nodes, the path must begin at one odd node and end at the other.

Pr: Let G have no odd nodes. Pick an arbitrary node m and begin an Euler path. Upon entering a new node, there always be an arc to exit through until we are back at m. If there is a node m' with unused arcs start an Euler cycle $m' \to m'$ and add it to the already constructed Euler path. This way we can make sure all the arcs are used.

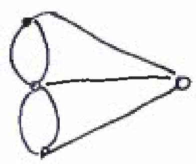
e.g. 

$(1 \to 2 \to 3 \to 4 \to 1) + (2 \to 5 \to 3 \to 6 \to 2)$

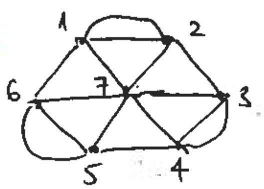
$(1 \to 2 \to 5 \to 3 \to 6 \to 2 \to 3 \to 4 \to 1)$

If there are two odd nodes we must start at one of them and end at the other. If the path has not covered all the arcs, extra cycles can be patched as above.

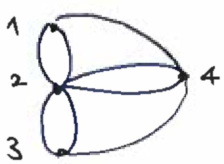
If there are more than two odd nodes, there could be no Euler path since odd nodes could be only initial or final points for such a path.

Ex: 

There is no Euler path in Königsberg.

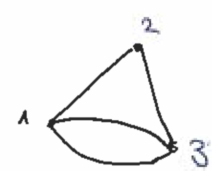
Ex: 

$1 \to 2 \to 1 \to 7 \to 2 \to 3 \to 7 \to 4 \to 3 \to 4 \to 5 \to 6 \to 5 \to 7 \to 6 \to 1$ is a Euler path.

Ex: 

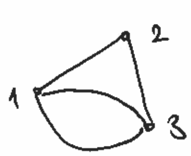
$1 \to 2 \to 1 \to 4 \to 2 \to 4 \to 3 \to 2 \to 3$ is a Euler path.

Ex: $G \to A = \begin{array}{c c} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} & \begin{array}{c} 3 \\ 2 \\ 3 \end{array} \end{array}$ Does it have an Euler path? Yes.



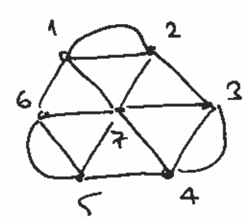
Def: A Hamiltonian circuit in a graph is a cycle using every node of the graph (recall that in a cycle only 1st = last node is repeated).

rem: There could be unused arcs, but there could be no repeated arcs since the endpoints will be repeated.

Ex: 

$1 \to 2 \to 3 \to 1$;

No. ;



$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 1$

Disc Math - § - Euler path and Hamiltonian circuit

rem: Determining whether Hamiltonian cycles exist in graphs is NP complete.

Euler path for comparison is $O(n^2)$.

rem: Suppose we are dealing with a weighted graph. If a Hamiltonian

TSP → circuit exists for the graph, can we find one with minimum weight?

The decision version of the travelling salesman problem, given a Hamiltonian circuit of length L is there a shorter Hamiltonian circuit for the graph is NP complete.

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4, 8, 20, 22, 26, 31, 33