§3. Disc Math - Quantifiers, Predicates, Validity

ex: All men are mortal. Socrates is a man ⊢ Socrates is mortal.
(can't determine validity with techniques explored so far).

ex: x is mortal → "is mortal" is predicate → predicate symbol $P(x)$.
y is Vanier student → predicate = "is Vanier student" → $Q(y)$.

Def: A property of a variable x, ranging over a set of possible values is called a predicate. The set of possible values of the variable x is its domain of interpretation.

ex: $P(n) \equiv$ The natural number n is prime.
domain $P(n) = \mathbb{N}$, truth set $\{n \in \mathbb{N} \mid P(n)\} = \{2, 3, 5, 7, 11, 13, \ldots\}$

ex: $P(x) = \{x^2 = x\}$ with domain $\mathbb{R}$.
$\{x \in \mathbb{R} \mid x^2 = x\} = \{0, 1\}$.

To change a predicate into a statement, we can:
① Assign specific values for the variables: $P(x) \to P(3)$.
② Add a universal or existential quantifier.

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not: $\forall$ - for all, universal quantifier.

Def: Let $Q(x)$ be a predicate with domain D. A universal statement is a statement of the form

$$\forall x \in D,\ Q(x) \qquad \text{(or } \forall x, Q(x) \text{ if the domain is understood)}$$

It is defined to be true iff the property $Q(x)$ holds for every $x \in D$.
Therefore it is false, if $Q(x)$ is false for at least one $x \in D$.

ex: $\forall x \in \mathbb{N}_0,\ x^2 \geq x$ - true; $\forall x \in \mathbb{R}_+,\ x^2 \geq x$ - false, $\because x = 0.2$.

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not: $\exists$ - there exist, existential quantifier

Def: Let $Q(x)$ be a predicate with domain D. An existential statement is a statement of the form $\exists x$ s.t. $Q(x)$. It is defined to be true if it is true for at least one $x \in D$. It is false if it is false for all $x \in D$.

ex: $\exists n \in \mathbb{N}$ s.t. $n^2 = n$ - true

$\exists x \in$ Pigs s.t. x could fly - false.

The relations between $\forall, \wedge$; $\exists, \vee$.

Let $P(x)$ be a predicate with finite domain with finite domain $D = \{x_1, .., x_n\}$

$\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \wedge .. \wedge P(x_n)$

$\exists x \in D$ s.t. $P(x) \equiv Q(x_1) \vee Q(x_2) \vee .. \vee Q(x_n)$.

ex: $D = \{0, 1\}$, $P(x) : \{x^2 = x\}$, $Q(x) = \{2x = x\}$.

$\forall x \in D, P(x) \equiv \{(0^2 = 0) \wedge (1^2 = 1)\} \equiv P(0) \wedge P(1)$ (true)

$\exists x \in D, Q(x) \equiv \{(2 \cdot 0 = 0) \vee (2 \cdot 1 = 1)\} \equiv Q(0) \vee Q(1)$ (true).

rem on notation: J. Gersting says: "A quantifier and its named variable are always placed in parenthesis" $\rightarrow (\forall x)(x > 0)$. Not really

We can have predicates which are binary or more generally n-ary and multiple quantifiers.

ex: $P(x, y) \equiv x \leq y$.

ex: $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}$ s.t. $x < y$ (There is a larger integer, true).

$\exists y \in \mathbb{N}$ s.t. $\forall x \in \mathbb{N}, x < y$ (There is a largest integer, false).

Moral: The order of the quantifiers is important!

rem: What is the difference between $\exists x \in D$ s.t. $\forall y \in E, P(x, y)$ and $\exists y \in D$ s.t. $\forall z \in E, P(y, z)$? $\rightarrow$ None! (x, y) ; (y, z) are dummy var's.

ex: True or false: $\forall x \in \mathbb{R}_+, \exists y \in \mathbb{R}_+$ s.t. $y^2 = x$ (True, $y = \sqrt{x}$)

(Stud). $\exists y \in \mathbb{R}_+$ s.t. $\forall x \in \mathbb{R}_+, y^2 = x$ (False!).

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Ex: We can have well-formed expressions with free variables → They are predicates:

$\forall x, P(x,y)$ (y is free).

Ex: $\forall x, [Q(x,y) \to \exists y \text{ s.t. } R(x,y)]$ (y is free).

$[\forall x, Q(x,y)] \to [\exists y \text{ s.t. } R(x,y)]$ (both x and y are free)

Negation: $\sim(\forall x \in D, P(x)) \equiv \exists x \in D \text{ s.t. } \sim P(x)$

$\sim(\exists x \in D \text{ s.t. } Q(x)) \equiv \forall x \in D, \sim Q(x)$

Ex: $\forall A \in M_{2,2}, (A^T = A) \to (A \text{ is invertible})$. False. Prove the negation:

$\sim(\forall A \in M_{2,2}, (A^T = A) \to (A \text{ is invertible})) \equiv$

$\equiv \exists A \in M_{2,2} \text{ s.t. } (A^T = A) \wedge (A \text{ is not invertible})$. True, e.g. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Translation:

Ex: a) Every parrot is ugly: $\forall x, P(x) \to U(x)$, $P(x) \equiv x$ is a parrot, $U(x) \equiv x$ is ugly; Domain of x is all birds, all animals, everything.

$\forall x, P(x) \wedge U(x)$ is an incorrect translation.

b) There is an ugly parrot: $\exists x \text{ s.t. } P(x) \wedge U(x)$. Now $\exists x \text{ s.t. } P(x) \to U(x)$

This last statement will be true as soon as there something which is not a parrot, i.e. $P(x)$ is false in this instance.

Ex: $D(x)$, x is a dog; $R(x)$, x is a rabbit; $C(x,y)$, x chases y.

a) All dogs chase all rabbits.

$\forall x, \forall y, D(x) \wedge R(y) \to C(x,y)$

Could also be written as: $\forall x, [D(x) \to \forall y, (R(y) \to C(x,y))]$

These two are equivalent since a quantifier can slide through a predicate that does not have the quantified variable and

$[(A \wedge B) \to C] \equiv [A \to (B \to C)]$.

b) Some dogs, chase all rabbits.

$\exists x, \forall y, [D(x) \wedge (R(y) \rightarrow C(x,y))]$.

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HW p50: 4, 7, 16, 22, 28, 34, 35
a, b, c, d

c) Only dogs chase rabbits.

$\forall x, \forall y, [R(y) \wedge C(x,y) \rightarrow D(x)]$.

Validity: The truth value of a propositional formula depends on the truth values assigned to the propositional letters in the formula. So if there are n letters there are 2^n rows in the truth table $\rightarrow 2^n$ interpretations. A propositional formula which true under every (of the 2^n interpretations) is a tautology.

The truth of a predicate formula depends on the interpretation, i.e. the assignment of domain to each variable and assignment of a property for each object in the domain for each predicate in the formula. The analogue of tautology for predicate formulas is validity - a predicate formula is (semantically) valid if it is valid in all interpretations (in all models). $P_1, \ldots, P_n \vdash q$ is a valid argument if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow q$ is a tautology

An algorithm exists to decide if a propositional formula is a tautology - construct the 2^n truth table. We clearly cannot examine all possible interpretations of a predicate formula; and there is no algorithm.

Gödel's 1st incompleteness theorem: Any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetic statement that is true (disquotational sense), but not provable. (No automatic proof verification is possible).

However we can check the validity (or lack of) for specific predicate formulas.

Ex: $\forall x, P(x) \rightarrow \exists x \text{ s.t. } P(x)$ is clearly true in every interpretation.

Ex: $\exists x \text{ s.t. } P(x) \rightarrow \forall x, P(x)$ is not valid; Ex: $\forall x, P(x) \rightarrow [Q(x) \rightarrow P(x)]$ is valid.