Disc Math - §4 - Predicate Logic

$P_1, P_2, \dots, P_n \vdash Q$, for this argument to be valid, $P_1 \wedge \dots \wedge P_n \to Q$ must be true in all interpretations. No equivalent of table is availabl[e]. We use again a system of derivation rules.

1. The deduction rules of propositional logic (both equivalence and inference) are still part of predicate logic.

Ex: MP is still valid $P, P \to Q \vdash Q$ but now it could read:

$$[\forall x, R(x)], [\forall x, R(x) \to \forall y, S(x)] \vdash \forall y, S(y)$$

2. And there are 4 new rules:

Universal Instantiation: $\forall x, P(x) \vdash P(t)$, t - variable or a constant

Existential Instantiation: $\exists a$ s.t. $P(x) \vdash P(a)$, a - constant (introduced by the rule

Universal Generalization: $P(x) \vdash \forall x, P(x)$, x - cannot be a free variable in a hypotes

Existential Generalization: $P(x) \vdash \exists x$ s.t. $P(x)$ or $P(a) \vdash \exists x$ s.t. $P(x)$.

Disclaimer: some restrictions on their use apply.

Ex: Universal instantiation: All humans a mortal. Socrates is huma[n]. Therefore, Socrates is mortal.

$H(x) \equiv x$ is human ; $M(x) \equiv x$ is mortal ; s - Socrates.

The argument is $[\forall x, H(x) \to M(x)], H(s) \vdash M(s)$ and here is the proof sequence:

$$\cfrac{\cfrac{\forall x, H(x) \to M(x)}{H(s) \to M(s)}\,\text{ui} \qquad H(s)}{M(s)}\,\text{MP}$$

Note that: $\forall x, \exists y$ s.t. $P(x,y) \vdash \exists y$ s.t. $P(y,y)$ would be invalid. E.g.

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $y > x$ is true, but

$\exists y \in \mathbb{R}$ s.t. $y > y$ is false. $\Rightarrow$ can't instantiate a variable bound by another quantifier.

Ex: Existential instantiation:

$$\dfrac{\dfrac{\forall x, P(x) \to Q(x)}{P(a) \to Q(a)} \qquad \dfrac{\exists y \text{ s.t. } P(y)}{P(a)}}{Q(a)}\ \text{MP}$$

← An element in the domain for which P(y) is true is now called a. This rule is used first.

Ex: Universal generalization:

$$\dfrac{\dfrac{\dfrac{\forall x, P(x)}{P(x)}\,\text{ui} \qquad \dfrac{\forall x, P(x) \to Q(x)}{P(x) \to Q(x)}\,\text{ui}}{Q(x)}\,\text{MP}}{\forall x, Q(x)}\,\text{ug}$$

x is a free variable, not a constant.

Note that:

$$\dfrac{\dfrac{\dfrac{\forall x, \exists y \text{ s.t. } Q(x,y)}{\exists y \text{ s.t. } Q(x,y)}\,\text{ui}}{Q(x,a)}\,\text{ei}}{\forall x, Q(x,a)}\,\text{ug}$$

« Invalid, can't universally generalize a variable in a term obtained by existential instantiation.

E.g.: $Q(x,y) \equiv (x=-y)$ $\quad \forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x=-y$ True

$\exists y \in \mathbb{R}$ s.t. $x=-y$ True ; $x=-4$ True ; $\forall x \in \mathbb{R}, x=-4$ False

Ex: Existential Generalization:

$$\dfrac{\dfrac{\forall x, P(x)}{P(x)}\,\text{ui}}{\exists x \text{ s.t. } P(x)}\,\text{eg}$$

Thus $\forall x, P(x) \vdash \exists x \text{ s.t. } P(x)$

Note that: $P(a,y) \vdash \exists y \text{ s.t. } P(y,y)$ is not valid.

E.g.: $P(x,y) \equiv (y>x)$, $a=0$ Then $P(0,y) \equiv \{y>0\}$ true in domain $\mathbb{N}$

$\exists y \in \mathbb{N}$ s.t. $y>y$ false.

Ex:

$$\dfrac{\exists x, P(x) \vee \exists x, Q(x)}{P(a) \vee Q(a)}$$

← Incorrect existential instantiation (shouldn't be a in both)

Ex: $\dfrac{\forall x, \exists y \text{ s.t. } Q(x,y)}{\forall x,\ Q(x,a)}$ ← incorrect ei. The $\exists$ quantifier should be left most.

rem: EI is the most peevish of the new rules.

Th: The set of derivation rules for Predicate Logic is correct and complete. We can prove only valid argument and every valid argument is provable using these rules. Validity is semi-decidable, but not decidable

Ex: Prove that the argument $\forall x, [P(x) \wedge Q(x)] \vdash \forall x, P(x) \wedge \forall x, Q(x)$ (redistributing and's!) is valid.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x, [P(x) \wedge Q(x)]}{P(x) \wedge Q(x)}\ \text{vi}}{P(x)}\ \text{sim}}{\forall x, P(x)}\ \text{vg} \qquad \dfrac{Q(x)}{\forall x, Q(x)}\ \text{vg}}{\forall x, P(x) \wedge \forall y, Q(y)}\ \text{con.}$$

rem: On the use of temporary hypothesis: If π is inserted into a proof tree and eventually W is deduced from π and the other hypothesis, then $\pi \to W$ has been deduced from the original hypothesis alone. [This is an extension of the deduction method].

Ex: The argument $[P(x) \to \forall y, Q(x,y)] \vdash \forall y, [P(x) \to Q(x,y)]$ is valid (can slide quantifiers).

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x)\ (\text{t.h.}) \qquad P(x) \to \forall y, Q(x,y)}{\forall y, Q(x,y)}\ \text{MP}}{Q(x,y)}\ \text{vi}}{P(x) \to Q(x,y)}\ (\text{discard t.h.})}{\forall y, P(x) \to Q(x,y)}\ \text{vg}$$

rem: $H \vdash P \to Q$ is valid when $H, P \vdash Q$ is.

Disc Math - §4 - Predicate Logic

Ex: $[\exists x \text{ s.t } P(x)]' \leftrightarrow \forall x, P(x)'$ is valid.

$\rightarrow$

$$\dfrac{\dfrac{\dfrac{P(x) \ (\text{t.h.})}{\exists x \text{ s.t. } P(x)}\,\text{eg}}{P(x) \rightarrow \exists x \text{ s.t. } P(x)}\,(\text{t.h. disch.}) \qquad [\exists x. \text{ s.t. } P(x)]'}{\dfrac{P(x)'}{\forall x, P(x)'}\,\text{ug}}\,\text{MT}$$

$\leftarrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists x \text{ s.t. } P(x) \ (\text{t.h.})}{P(a)}\,\text{ei} \qquad \dfrac{\forall x, P(x)'}{P(a)'}\,\text{ui}}{[\forall x, P(x)']'}\,\text{inconsistency}}{\exists x \text{ s.t. } P(x) \rightarrow [\forall x, P(x)']'}\,(\text{t.h.d.}) \qquad [[\forall x, P(x)']']' \ (\text{hyp. double neg})}{[\exists x \text{ s.t. } P(x)]'}\,\text{MT}$$

rem: We add the following two equivalences to the list of "known" rules:

$$[\exists x \text{ s.t } P(x)]' \equiv \forall x, P(x)'$$

$$[\forall x, P(x)]' \equiv \exists x \text{ s.t. } P(x)'$$

Ex: Is $(\exists x \text{ s.t. } P(x)) \wedge (\exists y \text{ s.t. } Q(y)) \vdash \exists z \text{ s.t. } P(z) \wedge Q(z)$ valid?

No! $D = \mathbb{R}$, $P(x)$ - positive, $Q(y)$ - negative.

Ex: Every CS student works harder than somebody. Everyone who works harder than another person gets to play less video games than the other person. Yannick is a CS student. Therefore Yannick plays less VG than someone else.

$$\forall x, \exists y \text{ s.t. } C(x) \rightarrow W(x,y),\ \forall x, \forall y, (W(x,y) \rightarrow G(x,y)),\ C(Y) \vdash$$

$$\vdash \exists z \text{ s.t. } G(Y, z)$$

Disc. Math - §4 - Predicate Logic

$$\dfrac{\forall x, \exists y \text{ s.t. } C(x) \to W(x,y)}{\exists y \text{ s.t. } C(Y) \to W(Y,y)}\ \text{ui}$$

$$\dfrac{\exists y \text{ s.t. } C(Y) \to W(Y,y)}{C(Y) \to \exists y \text{ s.t. } W(Y,y)}\ \text{equiv.}$$

$$\dfrac{C(Y) \qquad C(Y) \to \exists y \text{ s.t. } W(Y,y)}{\exists y \text{ s.t. } W(Y,y)}\ \text{MP}$$

$$\dfrac{\exists y \text{ s.t. } W(Y,y)}{W(Y,a)}\ \text{ei}$$

$$\dfrac{\forall x, \forall y, W(x,y) \to G(x,y)}{\forall y, W(Y,y) \to G(Y,y)}\ \text{ui} \qquad \dfrac{\forall y, W(Y,y) \to G(Y,y)}{W(Y,a) \to G(Y,a)}\ \text{ui}$$

$$\dfrac{W(Y,a) \qquad W(Y,a) \to G(Y,a)}{G(Y,a)}\ \text{MP}$$

$$\dfrac{G(Y,a)}{\exists z \text{ s.t. } G(Y,z)}\ \text{eg}$$

Ex: There is a town with one male barber. In this town every man keeps himself clean-shaven, by shaving himself or being shaved by the barber. The barber is a man in this town who shaves all those, and only those, men in town who do not shave themselves.

$M(x) \equiv x$ is a man in town; $S(x,y) \equiv x$ shaves y

$\exists x$ s.t. $\forall y, \left(M(x) \wedge M(y) \to \left(S(x,y) \leftrightarrow \sim S(y,y) \right) \right)$.

This is a contradiction.

Ex: Declarative languages → Prolog; Expert systems

Proof of correctness; proof of security.

HW §1.4 p69: 8, 12, 15,
18, 21, 24, 28, 34, 36, 38, 41