Disc Math - §6 - Induction

Th: $\sum_{j=1}^{n} j = 1+2+\dots+n = \frac{n(n+1)}{2}$

Pr: Base case $n=1$, $1 = \frac{1(1+1)}{2}$ ✓

suppose $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$, Prove That $\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}$

↑ Inductive hypothesis ↑ Inductive step

$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2} \quad \boxtimes$$

Method of proof by mathematical induction:

To prove a statement of the form : $\forall n \in \mathbb{Z}, n \geq a \to P(n)$:

1. Show $P(a)$ is true.
2. $\forall n \geq a$ show that $P(n) \to P(n+1)$.

(Then $P(a) \to P(a+1) \to P(a+2) \to \dots$ and we have $P(n)$ for all $n \in \mathbb{Z}, n \geq a$).

Th: $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ (Geometric sequence). $n = 0, 1, 2, \dots$

Pr: Base case $n=0$ $1 = \frac{1-r}{1-r}$ ✓.

Suppose $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$.

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1} = \frac{1-\cancel{r^{k+1}}+\cancel{r^{k+1}}-r^{k+2}}{1-r} = \frac{1-r^{k+2}}{1-r}$$

Ex: $\sum_{j=0}^{3} (1/2)^j = \frac{1-(1/2)^4}{1-1/2} = \frac{15}{8}$, $\sum_{j=0}^{9} (1/2)^j = \frac{-(1/2)^{10}+1}{1-1/2} = \frac{1023}{512}$, $\sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2$.

Ex: $\forall n \geq 0$, $2^{2n}-1$ is divisible by 3.

Pr: Base case $n=0$, $3 \mid 0$. ✓. Assume $3 \mid (2^{2k}-1)$.

$$2^{2(k+1)}-1 = 2^2 \cdot 2^{2k} - 1 = (3+1)2^{2k}-1 = 3 \cdot 2^k + (2^{2k}-1). \checkmark \boxtimes$$

Ex: Prove That $\prod_{k=2}^{n} (1 - 1/k) = \frac{1}{n}$, $n \geq 2$. Base case $(1-1/2) = 1/2$, ok.

Assume $\prod_{k=2}^{j} \left(1-\frac{1}{k}\right) = \frac{1}{j}$. Then $\prod_{k=2}^{j+1} \left(1-\frac{1}{k}\right) = \left[\prod_{k=2}^{j} \left(1-\frac{1}{k}\right)\right] \cdot \left(1-\frac{1}{j+1}\right) = \frac{1}{j}\left(1-\frac{1}{j+1}\right) = \frac{1}{j+1}$. $\boxtimes$

Prop: $\forall n \in \mathbb{N}$, $n \geq 3$, $2^n > 2n+1$.

Base case $n=3$: $2^3 = 8 > 2(3)+1 = 7$ o.k. Assume $2^k > 2k+1$.

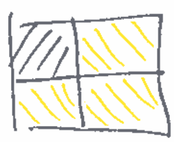
$2^{k+1} = 2 \cdot 2^k = 2^k + 2^k > (2k+1) + 2 = 2(k+1)+1$. ∎

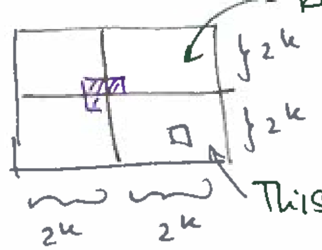
Ex: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $n \in \mathbb{Z}$, $n \geq 1$.

Base case trivial. Assume $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}. ∎$$

Ex: A checkerboard of size $2^n \times 2^n$ with one square removed can be tiled by L-shaped pieces: 

Pr: $n = 1$.  Assume $2^k \times 2^k$ checkerboard with one tile removed can be tiled. Consider $2^{k+1} \times 2^{k+1}$ checkerboard.



Remove the 3 corner squares, tile the quarters, tile the removed squares.

This quarter has a hole and can be tiled.

Principle of Strong Mathematical Induction: $P(n)$ predicate on $\mathbb{Z}$, $\mathbb{N}$ and let $a \leq b$.

① suppose $P(a), P(a+1), \ldots, P(b)$ are all true (base step).

② Suppose if for $k \geq b$ if $P(i)$ is true for all i: $a \leq i \leq k$ then $P(k+1)$ is true. We have that $P(n)$ is true $\forall n \geq a$.

Ex: Consider the sequence $\{s_n\}$, $n = 0, 1, 2, \ldots$ defined as follows:

$$s_0 = 0, \quad s_1 = 4, \quad s_k = 6s_{k-1} - 5s_{k-2}, \; k \geq 2$$

Show that $s_n = 5^n - 1$.

Pr: Basic case $s_0 = 5^0 - 1 = 0$, $s_1 = 5^1 - 1 = 4$ o.k. Now assume $s_i = 5^i - 1$, $0 \leq i \leq k$.

$s_{k+1} = 6s_k - 5s_{k-1} = 6(5^k - 1) - 5(5^{k-1} - 1) = (6-1)5^k - 1 = 5^{k+1} - 1$. ∎

Def: Principle of well-ordering for $\mathbb{N}$: every collection of positive integers that is nonempty has a smallest member.

Th: The following three principle are equivalent:
- principle of induction
- principle of strong induction
- well-ordering for $\mathbb{N}$. $\boxtimes$

rem: (WOP) Well-ordering principle: $\forall$ set can be well-ordered i.e. it could be endowed with linear order s.t. $\forall$ nonempty subset has a least element

→ transfinite induction

→ AC: If $\{A_i\}_{i\in I}$ is a family of nonempty sets, $\exists$ a function $f: I \to \bigcup_i A_i$ st. $f(i) \in A_i$, $\forall i \in I$ ($\exists$ choice function).

Ex: (part of FTA). $\forall n \in \mathbb{N}$ is a product of prime numbers or is a prime.

Pr: $n=2$ is prime. Assume that the statement is true for $k \geq 2$. Now consider $k+1$. If $k+1$ is a prime we are done. If not $k+1 = ab$ with $2 \leq a, b \leq k$. By the inductive assumption a, b are each either prime or product of primes. So $k+1$ is a product of primes. $\boxtimes$

HW §2.2 p 122

4, 8, 18, ~~22~~, 28, 34, 40, 48

56, 67, 72b