Disc Math - §9 - Recurrence Relations.

Linear first-order recurrence relations.

Ex: The sequence defined by $S(1)=1$, $S(n+1)=3S(n)$. It starts
$1, 3, 9, 27, \dots$ A guess for the general term is $S(n)=3^{n-1}$.
Proof by induction: $n=1$ $3^{1-1}=3^0=1$ ✓ Assume $S(n)=3^{n-1}$.
$S(n+1)=3\cdot 3^{n-1}=3^n=3^{(n+1)-1}$. ✓ ▨

rem: $S(n)=3^{n-1}$ is a closed form solution of the above recurrence rela

Def: A recurrence relation for a sequence $S(n)$ is linear if it is of the f

$$S(n)=f_1(n)S(n-1)+\dots+f_k(n)S(n-k)+g(n);$$

This relation is of k'th order. The recurrence relation is homogene
if $g(n)=0$ $\forall n$.

constr: Let's solve the general 1^{st} order recurrence relation with constant coefficients: $S(n)=cS(n-1)+g(n)$:

$$S_n = cS(n-1)+g(n) = c\left(cS(n-2)+g(n-1)\right)+g(n) =$$
$$= c^2S(n-2)+cg(n-1)+g(n) = c^2\left(cS(n-3)+g(n-2)\right)+cg(n-1)+g(n$$
$$= c^3S(n-3)+c^2g(n-2)+cg(n-1)+g(n) = \dots$$
$$S(n) = c^{n-1}S(1)+c^{n-2}g(2)+\dots+cg(n-1)+g(n)$$
$$\boxed{S(n)=c^{n-1}S(1)+\sum_{i=2}^{n}c^{n-i}g(i)}$$

rem: Of course we would like to have a closed formula for the sum

Ex: $S(n)=3S(n-1)+2$, $n\geq 2$ $S(1)=1$.

$$S(n)=3^{n-1}S(1)+\sum_{i=2}^{n}3^{n-i}(2) = 3^{n-1}+2\sum_{i=1}^{n}3^{n-i}-2\cdot 3^{n-1}$$
$$=2\cdot 3^n\sum_{i=1}^{n}\left(\frac{1}{3}\right)^i-3^{n-1} = 2\cdot 3^n\,\frac{1/3-(1/3)^{n+1}}{1-1/3}-3^{n-1} = \cancel{2\cdot 3^n\,\frac{3}{3^{n+1}}}\ \cancel{\frac{3^n-1}{3-1}} -$$
$$=3^n-3^{n-1}-1$$

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ex: Solve the recurrence relation $T(n) = T(n-1) + n$; $T(1) = 1$

$$T(n) = 1^{n-1}\, T(1) + \sum_{i=2}^{n} (1)^{n-i}\, i = 1 + \sum_{i=2}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+}{2}$$

confirmation by induction $T(1) = \frac{1(2)}{2} = 1$ ✓

Assume $T(k) = \frac{k(k+1)}{2}$. Then $T(k+1) = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} = \frac{(k+1)(k}{2.}$

ex: (Non-constant coefficients). $T(1) = 1$; $T(n) = 2n\, T(n-1)$, $n \geq 2$.

$$T(n) = 2n\, T(n-1) = 2n\left[2(n-1)\, T(n-2)\right] = 2^2 n(n-1)\, T(n-2) = \dots =$$

$$= 2^k n(n-1)\dots(n-(k-1))\, T(n-k) \to$$

$$T(n) = 2^{n-1} n(n-1)\dots 2\, T(1) = 2^{n-1} n!$$

Verify by induction: Base case: $T(1) = 2^0 1! = 1$. Assume $T(k) = 2$

$T(k+1) = 2(k+1)\, T(k) = 2(k+1)\, 2^{k-1} k! = 2^k (k+1)!$. ✓ ☒

Linear Second Order Recurrence Relations.

* The second order homogeneous recurrence relation with constant co

$$S(n) = c_1 S(n-1) + c_2 S(n-2)$$

e.g. Fibonacci sequence $F(n) = F(n-1) + F(n-2)$, $n > 2$; $F(1) = 1 = F(2)$.

Def: $t^2 - c_1 t - c_2 = 0$ is called the characteristic equation of the above r.r

constr: Ansatz: a closed form solution is of the form:

$$S(n) = p\, r_1^{n-1} + q\, r_2^{n-1}$$

where r_1, r_2 are the two distinct roots of the characteristic equatio

The coefficients p, q are chosen to satisfy the initial conditions:

$$S(1) = p\, r_1^{1-1} + q\, r_2^{1-1} = p + q$$

$$S(2) = p\, r_1^{2-1} + q\, r_2^{2-1} = p r_1 + q r_2$$

Proof: By induction. The base cases $S(1), S(2)$ are settled by the initia condition equations above. Assume $S(i) = p\, r_1^{i-1} + q\, r_2^{i-1}$, $1 \leq i \leq k$. Must prove $S(k+1) = p\, r_1^{k} + q\, r_2^{k}$.

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Notice $r_1^2 = c_1 r_1 + c_2$; $r_2^2 = c_1 r_2 + c_2$. Now,

$$S(k+1) = c_1 S(k) + c_2 S(k-1) = c_1(p r_1^{k-1} + q r_2^{k-1}) + c_2(p r_1^{k-2} + q r_2^{k-2})$$
$$= p r_1^{k-2}(c_2 + c_1 r_1) + p r_2^{k-2}(c_2 + c_1 r_2) = p r_1^{k-2} r_1^2 + q r_2^{k-2} r_2^2 =$$
$$= p r_1^k + q r_2^k. \quad \blacksquare$$

Ex: Solve the recurrence relation: $S(n) = 2S(n-1) + 3S(n-2)$, $n \geq$ subject to $S(1) = 3$, $S(2) = 1$.

Sol: $t^2 - 2t - 3 = 0$ $\quad r_1 = 3 \quad r_2 = -1$. $\quad S(n) = p\, 3^{n-1} + q(-1)^{n-1}$

$p + q = 3$, $p(3) + q(-1) = 1 \Rightarrow p = 1, q = 2 \Rightarrow S(n) = 3^{n-1} + 2(-1)^{n-}$

Ex: Solve the Fibonacci relation: $F(n+2) = F(n+1) + F(n)$, $F(1) = F(2) = 1$

$t^2 - t - 1 = 0 \quad t_{1,2} = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

$F(n) = p\left(\dfrac{1+\sqrt{5}}{2}\right)^{n-1} + q\left(\dfrac{1-\sqrt{5}}{2}\right)^{n-1}$, $\quad p = \dfrac{1}{\sqrt{5}}\dfrac{1+\sqrt{5}}{2}$, $q = \dfrac{-1}{\sqrt{5}}\dfrac{1-\sqrt{5}}{2}$.

$F(n) = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

Constr: If the characteristic equation $t^2 - c_1 t - c_2 = 0$ has a double r $r_1 = r_2 = r$, the solution for the recurrence relation takes the form

$$S(n) = p r^{n-1} + q(n-1) r^{n-1}, \ n \geq 1; \quad p = S(1), \quad pr + qr = S(2).$$

Ex: Solve the recurrence relation

$$S(n) = 8S(n-1) - 16S(n-2), \ n \geq 3; \quad S(1) = 1, \ S(2) = 12.$$

Sol: $t^2 - 8t + 16 = 0 \quad (t-4)^2 = 0$, $r = 4$

$S(n) = p\, 4^{n-1} + q(n-1) 4^{n-1}$, $\quad p = 1 \quad p(4) + q(4) = 12$

$S(n) = 4^{n-1} + 2(n-1)4^{n-1} = (2n-1)4^{n-1}$. $\blacksquare$

Divide-and-conquer recurrence relations.

$$S(n) = c\, S\left(\frac{n}{2}\right) + g(n), \quad n \geq 2, \ n = 2^m$$

Solution is:

$$S(n) = c^{\log_2 n} S(1) + \sum_{i=1}^{\log_2 n} c^{(\log_2 n - i)} g(2^i)$$

Prove in the book by a reduction to 1st order cas

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Ex: solve the recurrence relation

$$T(n) = 2T\left(\frac{n}{2}\right) + 2n, \quad T(1) = 3. \qquad n = 2^k!$$

Sol: Here $c = 2$, $g(2^j) = 2(2^j) \Rightarrow$

$$T(n) = 2^{\log_2 n} T(1) + \sum_{i=1}^{\log_2 n} 2^{\log_2 n - i}\, 2(2^i) =$$

$$= n(3) + \sum_{i=1}^{\log_2 n} 2^{\log_2 n + 1} = 3n + \left(2^{\log_2 n + 1}\right)\log_2 n =$$

$$= 3n + 2n\log_2 n. \quad \boxtimes$$

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3, 6, 10, 18, ~~22~~, 28, 32, 38, 48.