Disc Math - H15 - Test3 - solutions

① a) $\sim(\exists (x,y)\in \mathbb{Z}^2 \text{ s.t. } x>1 \wedge y>1 \wedge xy=23\} = \forall (x,y)\in\mathbb{Z}^2,\ x\le 1 \vee y\le 1 \vee xy\neq$

Original is false since 23 is prime number.

b) $\sim(\forall x\in\mathbb{R}, \exists y\in\mathbb{R} \text{ s.t. } xy\ge 1) = \exists x\in\mathbb{R} \text{ s.t. } \forall y\in\mathbb{R},\ xy<1$

Negation is true $x=0$.

② Prove the contrapositive, $\forall x\in\mathbb{R},\ x\in\mathbb{Q}\to x^4\in\mathbb{Q}$. Let $x=\frac{p}{q}$. Then $q\neq 0$,
$\left(\frac{p}{q}\right)^4 = \frac{p^4}{q^4}\in\mathbb{Q}$.

③ Let $n=p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$. Then $n^m = p_1^{mn_1}p_2^{mn_2}\cdots p_k^{mn_k}$. We have

$$\varphi(n^m) = p_1^{mn_1-1}p_2^{mn_2-1}\cdots p_k^{mn_k-1}\varphi(p_1)\varphi(p_2)\cdots\varphi(p_k) =$$

$$= \frac{p_1^{mn_1}p_2^{mn_2}\cdots p_k^{mn_k}}{p_1p_2\cdots p_k}\left(\frac{\varphi(n)}{p_1^{n_1-1}p_2^{n_2-1}\cdots p_k^{n_k-1}}\right) = \frac{p_1^{mn_1}p_2^{mn_2}\cdots p_k^{mn_k}}{p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}}\varphi(n) = \frac{n^m}{n}\varphi(n) =$$

$$= n^{m-1}\varphi(n).$$

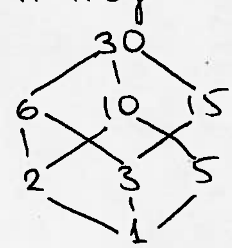
④ By contradiction: Let $f: S\to P(S)$ be a bijection. $\forall s\in S,\ f(s)\in P(S)$. Define
$X=\{x\in S \mid x\notin f(x)\}$. Since $X\in P(S)$, $\exists y\in S$ s.t. $f(y)=X$.

Now, if $y\in X$ then $y\notin f(y)=X$.
On the other hand if, $y\notin X$ then $y\in X=f(y)$.
} contradiction.

Therefore, there is no bijection $S\to P(S)$.

⑤ Hands missing at least one suit: ${}_4C_1\cdot{}_{39}C_5 - {}_4C_2\cdot{}_{26}C_5 + {}_4C_3\cdot{}_{13}C_5$. Hands not missing a suit ${}_{52}C_5 - {}_4C_1\cdot{}_{39}C_5 + {}_4C_2\cdot{}_{26}C_5 - {}_4C_3\cdot{}_{13}C_5 = 2598960 - 230302$
$+ 394680 - 5148 = 685464$

⑥ Reflexive: $x|x$. Antisymmetric $x|y \wedge y|x \Rightarrow y=x$. Transitive $x|y \wedge y|z \Rightarrow x|z$.



⑦ In each step of the of the decimal expansion we register the remainder. In the next step we multiply the remainder by 10 and divide again.

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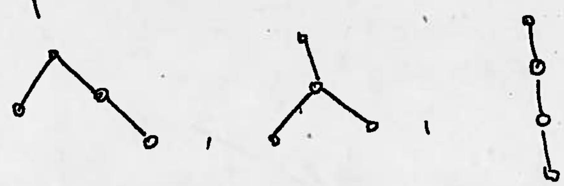
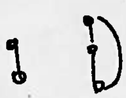
Since there are 6 remainders available the expansion will repeat after at most 6 divisions.

(8) a) $y = x + kn$, $w = z + ln$; $yw = (x+kn)(z+ln) = xz + n(kz + lx + kln) \Rightarrow$
$yw \equiv xz \pmod n$

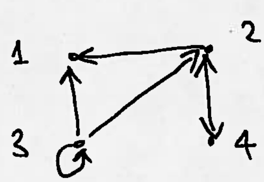
b) $y = x + kn$, $y^s = (x+kn)^s = x^s + \sum_{i=1}^{s} x^{s-i}(kn)^i = x^s + k\sum_{i=1}^{s} x^{s-i} k^{i-1} n^i \Rightarrow$
$y^s \equiv x^s \pmod n$.

(9) f is onto; $\forall A \in \mathcal{P}(\mathbb{R})$, $A = f((A, \mathbb{R}))$; f is not 1-1, indeed $f((A,A)) = f((A,\mathbb{R}$
Consequently f is not a bijection.
h is onto: $h((\mathbb{R}\setminus A)) = A$; h is a bijection, $h = h^{-1}$, i.e. $h(h(A)) = (A')' = A$
In particular h is 1-1.

(10) 

(11) a) $K'_{2,3}$ 

b) If G is not connected then G consists of two or more connected sub graphs that have no paths between them. Let x and y be distinct nodes. If x, y are in different subgraphs there is $x-y$ arc in G' and hence a path from x to y in G'. If x and y are in the same subgraph, then pick a node z in a different subgraph. Then $x-z-y$ is a path from x to y in G'.

(12) 

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad \bar{\rho} = \{(2,1), (2,2), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,4)\}.$$

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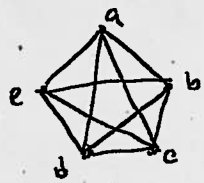
(13) a)

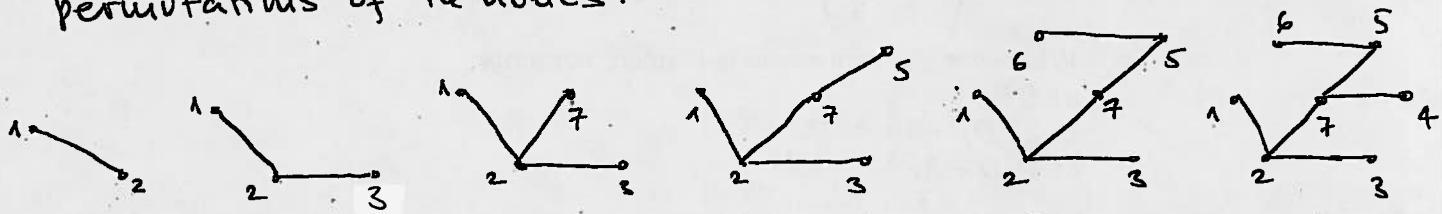


Hamiltonian cycle : a-b-c-d-e-a

Euler cycle : a-b-c-d-e-a-c-e-b-d-a

b) $n>2$. There are $n!$ Hamiltonian cycles for K_n, $n>2$ corresponding to the permutations of the nodes.

(14)



(15) a) Starting with a node x_1 in the set of m nodes, the algorithm visits next all nodes $y_1, \dots, y_n$ in the set of n nodes. After that, it will visit the remaining nodes in the set of m nodes.

b) Starting with node x_1 in the set of m nodes, the algorithm visit nodes alternating from the two sets: $x_1, y_1, x_2, y_2, \dots$ until one of the two sets has n unvisited nodes left and then visits the remaining nodes.

(16) a) $x(z+y)+(x'+y)' = x(z+y)+x\cdot y' = x(z+y+y') = x(z+1) = x\cdot 1 = x$

b) Let $x\cdot y'=0$. Then $xy = xy+0 = xy+xy' = x(y+y') = x\cdot 1 = x$

Let $xy=x$. Then $xy' = (xy)y' = x(yy') = x0 = 0$

(17) $f = ab'cd + a'b'cd + a'bcd + abc'd + ab'c'd + a'b'c'd + a'bc'd + abcd'$

	ab	ab'	a'b'	a'b
cd		1	1	1
cd'	1			
c'd'				
c'd	1	1	1	1

$f = (a'+b'+c')d + abcd'$