DISCRETE MATHEMATICS, H15, FINAL EXAMINATION

(1) (2 marks) Write the negations of the following two statements:
a) $\exists (x, y) \in \mathbb{Z}^2$ such that $x > 1 \wedge y > 1 \wedge xy = 23$.
b) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $xy \geq 1$.
For a), b) argue that the original statement is true or that the negation is true.

(2) (2 marks) Prove or disprove that for all $x \in \mathbb{R}$, if x^4 is irrational, then x is irrational.

(3) (2 marks) Prove that for m and n positive integers, $\phi(n^m) = n^{m-1}\phi(n)$.

(4) (2 marks) Prove Cantor's Theorem: For any set S, S and the power set $\mathcal{P}(S)$ have different cardinalities.

(5) (2 marks) How many poker hands contain at least one card in each suit ($\spadesuit, \heartsuit, \clubsuit, \diamondsuit$)?

(6) (2 marks) Consider the set $S = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and the relation ρ on S defined by $x \rho y \leftrightarrow x|y$.
a) Show that ρ is a partial order on S.
b) Draw the corresponding Hasse diagram.

(7) (1.5 marks) Show that in the decimal expansion of the quotient of two integers, eventually some block of digits repeats, e.g. $217/660 = 0.32\overline{87}\ldots$. (Hint: When we divide a by b, the possible remainders are $0, 1, \ldots, b-1$).

(8) (2 marks) Assume that $x \equiv y(\bmod n)$ and $z \equiv w(\bmod n)$ for some positive integer $n > 1$. Prove that
a) $x \cdot z \equiv y \cdot w(\bmod n)$,
b) $x^s \equiv y^s(\bmod n)$ for $s \in \mathbb{N}$.

(9) (2 marks) Define the functions $f : \mathcal{P}(\mathbb{R})^2 \to \mathcal{P}(\mathbb{R})$ by $f((A, B)) = A \cap B$ and $h : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $h(A) = A'$. Are the functions f, h onto, one-to-one, bijections? Justify your answers.

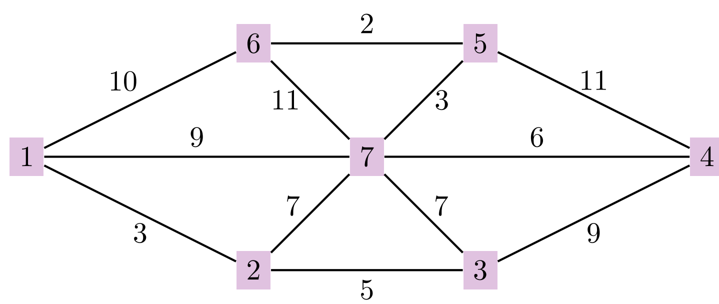
(10) (1 mark) Draw all nonisomorphic rooted binary trees with four nodes.

(11) (2 marks) If G is a simple graph, the complement of G, denoted G', is the simple graph with the same set of nodes as G, where nodes x, y are adjacent in G' iff they are not adjacent in G.
a) Draw the complement of $K_{2,3}$.
b) Prove that for a simple graph G with at least two nodes, if G is not connected, then G' is connected.

(12) (2 marks) Consider the binary relation $\rho = \{(2,1), (2,4), (3,1), (3,2), (3,3), (4,2)\}$ on the set $\{1, 2, 3, 4\}$.
a) Draw the associated directed graph and the adjacency matrix.
b) Determine the transitive closure of ρ by computing the reachability matrix (show details).

(13) (2 marks) a) Give an example of a simple graph that has an Euler cycle and a Hamiltonian cycle that are not identical.
b) For what values of n does a Hamiltonian cycle exists in K_n? For those values of n for which there is a Hamiltonian cycle in K_n how many distinct Hamiltonian cycles are there in K_n? Explain.

(14) (1.5 marks) For the weighted graph below, while describing every step in the algorithm you are using, find a minimal spanning tree.



(15) (2 marks) Describe the order in which nodes are visited both for a) breadth-first and b) for depth-first search of the bipartite complete graph $K_{m,n}$.

(16) (2 marks) Prove the following properties of Boolean algebras. Give a reason for each step.

$$a) \quad x \cdot (z + y) + (x' + y)' = x$$
$$b) \quad \{x \cdot y' = 0\} \leftrightarrow \{x \cdot y = x\}.$$

(17) (2 marks) Design a logical circuit with four binary inputs a, b, c, d, that computes the expression $(abc+d)$ in $\mathbb{Z}_2$. Find a truth function and a DNF for this expression. Simplify the DNF as much as possible and draw the associated logical network.