Disc Math - H16 - Test 3 - Solutions. ①

① Reflexivity: $5 \mid (m^2-m^2) \Leftrightarrow 5 \mid 0$ ✓

Symmetry: $5 \mid (m^2-n^2) \rightarrow 5 \mid (n^2-m^2)$ since $(n^2-m^2) = -(m^2-n^2)$ ✓

Transitivity: $5 \mid (m^2-n^2)$ and $5 \mid (n^2-p^2) \rightarrow 5 \mid (m^2-p^2)$ since
$m^2-p^2 = (m^2-n^2)+(n^2-p^2)$.

Thus D is an equivalence relation.

If $5 \mid m-n$ then $5 \mid (m^2-n^2)$. Thus the equivalence classes of D contai
the equivalence classes of mod 5. But also $3^2-2^2=5$ and $4^2-1^2=15$.
The equivalence classes of D are $[0], [1], [2]$.

② Reflexivity: $(a,b)\,\tau\,(a,b)$ since $a=a$ and $b\sigma b$.

Antisymmetry: $(a_1,b_1)\,\tau\,(a_2,b_2)$ and $(a_2,b_2)\,\tau\,(a_1,b_1)$ when $a_1 \neq a$
$\Rightarrow a_1 \rho a_2 \wedge a_2 \rho a_1 \Rightarrow a_1 = a_2$, i.e. this cannot happen.
$(a,b_1)\,\tau\,(a,b_2)$ and $(a,b_2)\,\tau\,(a,b_1) \Rightarrow b_1 \sigma b_2 \wedge b_2 \sigma b_1 \Rightarrow b_1 = b_2$.

Transitivity: Let $(a_1,b_1)\,\tau\,(a_2,b_2)$ and $(a_2,b_2)\,\tau\,(a_3,b_3)$.

Case 1: $a_1 \neq a_2$ or $a_2 \neq a_3$. Then $a_1 \neq a_3$ and $a_1 \rho a_3$ by the transitivi
of ρ and hence $(a_1,b_1)\,\tau\,(a_3,b_3)$.

Case 2: $a_1 = a_2 = a_3$. Then $b_1 \sigma b_2 \wedge b_2 \sigma b_3 \Rightarrow b_1 \sigma b_3$ by the transi
tivity of σ. Hence $(a_1,b_1)\,\tau\,(a_3,b_3)$ and τ is transitive.

③ a) By contradiction. If $x,y \in S$ s.t. $f(x)=f(y)$ then $(g\circ f)(x) = (g \circ f)(y)$.
But this contradicts the fact that $g\circ f$ is an injection. Thus f is an injecti

b) By contradiction. Let $\exists z \in U$ s.t. $z \notin im(g)$. Then $z \notin im(g\circ f)$. Contradic
tion. Thus g is a surjection.

c)

$S \xrightarrow{f} T \xrightarrow{g} U$

Here $g\circ f$ is a bijection, but neither f nor g are bijections.

④ a) $$\frac{22!}{9!\,8!\,2!\,3!} = 6401795400 \sim 6.4 \times 10^{10}$$

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(b) ${}_8C_6 \cdot {}_{11}C_6 + {}_8C_7 \cdot {}_{11}C_5 + {}_8C_8 \cdot {}_{11}C_4 = (28)(462) + (8)(462) + 330 = 16962$

(c) ${}_{12}P_5 = \dfrac{12!}{7!} = 95040$

(5) $12x + 24 = 48 + 39x$ in Z_{53}. $27x = -24$, $27x = 29$, $x = [27]^{-1} 29$

$53 = 1 \cdot 27 + 26$, $27 = 1 \cdot 26 + 1$, $1 = 27 - 26 = 27 - (53 - 27) = 2 \cdot 27 - 53$

$[27]^{-1} = 2$, $x = 2 \cdot 29 = 58 = 5$ in Z_{53}.

(6)

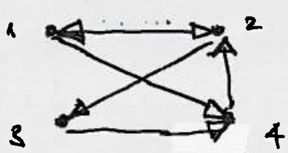
$n = 5$, $a = 6$, $r = 3$

$5 - 6 + 3 = 2$

(7) a) Such a graph is a tree. It will have $n-1$ edges.

b) This is K_n. It has $n(n-1)/2$ edges.

(8)



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad A^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$R = A \vee A^{(2)} \vee A^{(3)} \vee A^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The transitive closure of ρ is the full relation on S.

(9) Let H be the Hamiltonian circuit in G. It is a subgraph of G. To disconnect G we have to disconnect H. To disconnect a cycle we need to remove two nodes. Thus $\kappa(H) = 2$ and $\kappa(G) \geq 2$.

10 Organize the c components of G in a sequence and connect the neighbours in the sequence with one new arc for each neighbouring pair. Connect also the first component and the last component of the sequence. c new arcs are needed to do this and the resulting graph has Euler cycle. The two new arcs reaching a component have to be attached to the same node.

11 $P=\{1\}$

	1	2	3	4	5	6	7
d	0	3	∞	∞	∞	10	∞
pred	–	1	–	–	–	1	–

$P=\{1,2\}$

	1	2	3	4	5	6	7
d	0	3	8	∞	∞	10	11
pred	–	1	2	–	–	1	2

$P=\{1,2,3\}$

	1	2	3	4	5	6	7
d	0	3	8	28	10	10	11
pred	–	1	2	3	3	1	2

$P=\{1,2,3,5\}$

	1	2	3	4	5	6	7
d	0	3	8	17	10	10	11
pred	–	1	2	5	3	1	2

$P=\{1,2,3,5,7\}$

	1	2	3	4	5	6	7
d	0	3	8	17	10	10	11
pred	–	1	2	5	3	1	2

$P=\{1,2,3,4,5,7\}$

	1	2	3	4	5	6	7
d	0	3	8	17	10	10	11
pred	–	1	2	5	3	1	2

The shortest path is

1-2-3-5-4

and the shortest distance is 17.

12 By contradiction. Suppose that T is a minimal spanning tree that does not include the arc a. Add a, which then will be part of a cycle. Remove a different, higher weight arc from this cycle. This results

In a new spanning tree with lower weight and contradicts the fact that the original spanning tree had minimal weight.

(13) $(A\cap B)\setminus(B\cap C) \underset{def}{=} (A\cap B)\cap(B\cap C)' \underset{D.M.}{=} (A\cap B)\cap(B'\cup C') \underset{distr.}{=}$

$= ((A\cap B)\cap B')\cup((A\cap B)\cap C') \underset{assoc.}{=} (A\cap(B\cap B'))\cup((A\cap B)\cap C') =$

$\underset{compl.}{=} (A\cap\emptyset)\cup((A\cap B)\cap C') \underset{abs.}{=} \emptyset\cup((A\cap B)\cap C') \underset{id.}{=} (A\cap B)\cap C' \underset{def}{=} (A\cap B)\setminus C$

(14) a) $p \underset{ident.}{\equiv} p\vee c \equiv p\vee(p\vee q) \underset{assoc.}{\equiv} (p\vee p)\vee q \underset{idemp.}{\equiv} p\vee q \equiv c$.

Similarly $q\equiv c$.

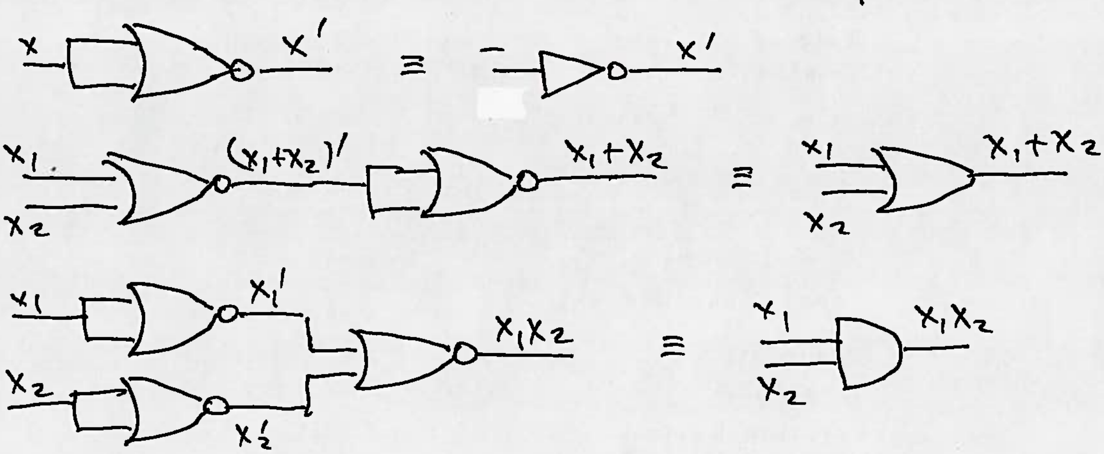
b) If $p\equiv q$ then $(p\wedge q')\vee(q\wedge p') \equiv (p\wedge p')\vee(p\wedge p') \underset{compl.}{\equiv} c\vee c \underset{idemp.}{\equiv} c$

If $(p\wedge q')\vee(q\wedge p')\equiv c$ then by a) $p\wedge q'\equiv c$ and $q\wedge p'\equiv c$.

We have $p \underset{id}{\equiv} p\wedge t \underset{compl.}{\equiv} p\wedge(q\vee q') \underset{distr.}{=} (p\wedge q)\vee(p\wedge q') \equiv (p\wedge q)\vee c \equiv$

$\equiv (p\wedge q)\vee(q\wedge p') \underset{distr}{\equiv} (p\vee p')\wedge q \underset{compl.}{\equiv} t\wedge q \underset{id}{\equiv} q$.

(15)



(16)

	x_1x_2	x_1x_2'	$x_1'x_2'$	$x_1'x_2$
x_3x_4	1	1	1	1
x_3x_4'		1		
$x_3'x_4'$		1		1
$x_3'x_4$	1	1	1	1

$f = x_4 + x_1x_2' + x_1'x_2x_3'$