DISCRETE MATHEMATICS, H16, TEST 3

(1) (2.5 marks) Let D be the relation defined on $\mathbb{Z}$ as follows: $\forall m, n \in \mathbb{Z}$

$$\{m\ D\ n\} \leftrightarrow 5|(m^2 - n^2)$$

Prove that this relation is an equivalence relation and describe the distinct equivalence classes of this relation.

(2) (2.5 marks) Let ρ be a partial order on a set A and let σ be a partial order on a set B. Define a relation τ on the Cartesian product $A \times B$ by

$$(a_1, b_1)\tau(a_2, b_2) \leftrightarrow \{(a_1 \neq a_2 \wedge a_1 \rho a_2) \vee (a_1 = a_2 \wedge b_1 \sigma b_2)\}.$$

τ is called lexicographic order on the Cartesian product $A \times B$. Prove that τ is also a partial order.

(3) (2 marks) Let $f : S \to T$ and $g : T \to U$ be functions. Prove or disprove the following assertions:
a) If $g \circ f$ is an injection, so is f.
b) If $g \circ f$ is a surjection, so is g.
c) If $g \circ f$ is a bijection, so are both f and g.

(4) (2 marks) Joyce is the head of the software solutions department at the Canadian Impudent Bank of Confusion.

a) Joyce has on staff 22 programmers who can code in C++. In how many ways can she assign them to 4 different team C++ programming tasks if the first task requires 9 programmers, the second 8 programmers, the third 2 programmers and the fourth 3 programmers?

b) Joyce has 19 programmers who are fluent in Java of which 8 are very experienced. In how many ways can she form a team of 12 programmers of which more than 5 are very experienced?

c) Joyce also manages 12 JavaScript Web developers. In how many ways can she select 5 JavaScript developers for 5 different tasks?

(5) (2 marks) Solve the equation $12x + 24 = 48 + 39x$ in $\mathbb{Z}_{53}$.

(6) (1.5 marks) Prove that $K_{2,3}$ is a planar graph. Confirm Euler's formula for $K_{2,3}$.

(7) (1.5 marks) For the following questions write at least one sentence in support of your answer quoting known properties of graphs.

a) What is the minimum number of edges that a simple connected graph with n vertices could have?

b) What is the maximum number of edges that a simple graph with n vertices could have?

(8) (2 marks) Consider the binary relation $\rho = \{(1,2),(1,4),(2,1),(2,3),(3,4),(4,2)\}$ on the set $S = \{1,2,3,4\}$.

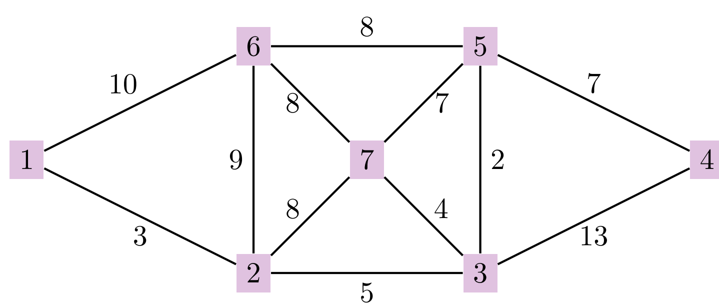
a) Draw the associated directed graph and the adjacency matrix.

b) Determine the transitive closure of ρ by computing the reachability matrix (show details).

(9) (1.5 marks) Let G be a graph. The connectivity of G, denoted by $\kappa(G)$, is the minimum number of nodes whose removal results in either disconnected graph or a single node. Prove that if a graph G has a Hamiltonian circuit then $\kappa(G) \geq 2$.

(10) (1.5 marks) Suppose that a graph G has c components and each node has even degree. What is the minimum number of arcs that must be added to G to obtain a graph with an Euler cycle? Does it matter where the new arcs are attached? Explain.

(11) (2 marks) For the weighted graph below, while describing every step in the algorithm you are using, find the shortest distance between node 1 and node 4.



(12) (1.5 marks) Let a be the arc of lowest weight in a weighted graph. Show that a must be an arc in any minimal spanning tree.

(13) (2 marks) Use Boolean algebra identities to prove that for any three sets A, B, C,

$$(A \cap B) \backslash (B \cap C) = (A \cap B) \backslash C.$$

Cite the identity you are using at each step.

(14) (2.5 marks) Use Boolean algebra identities to prove that in propositional logic:

a) If $p \vee q \equiv c$, then $p \equiv c$ and $q \equiv c$, where c stands for a contradiction.

b) $p \equiv q$ if and only if $(p \wedge q') \vee (q \wedge p') \equiv c$.

(15) (2.5 marks) A NOR gate receives inputs x_1 and x_2, where x_1 and x_2 are bits and produces output denoted $x_1 \downarrow x_2$, where

$$x_1 \downarrow x_2 = \begin{cases} 0 & \text{if } x_1 = 1 \text{ or } x_2 = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Draw three logical circuits which show that the AND, OR and NOT gates can be simulated with NOR gates.

(16) (2.5 marks) Consider the truth function

x_1	x_2	x_3	x_4	$f(x_1, x_2, x_3, x_4)$
1	1	1	1	1
1	1	1	0	0
1	1	0	1	1
1	1	0	0	0
1	0	1	1	1
1	0	1	0	1
1	0	0	1	1
1	0	0	0	1
0	1	1	1	1
0	1	1	0	0
0	1	0	1	1
0	1	0	0	1
0	0	1	1	1
0	0	1	0	0
0	0	0	1	1
0	0	0	0	0

Write and then maximally simplify the corresponding Boolean expression. Draw a minimal Logical Network which computes this truth function using AND, OR and NOT gates.