DISCRETE MATHEMATICS, A24, TEST 3

Name: __________________ Student number__________________

(1) (2 marks) For the two statements bellow write the negation and then present an argument that original statement is true or that the negation is true.
i) $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ s.t. $x + y > 0$.
ii) $\exists x \in \mathbb{N}$ s.t. $\forall y \in \mathbb{Z}\ x + y > 0$.

i) Negation $\exists x \in \mathbb{Z}$ s.t. $\forall y \in \mathbb{Z}$, $x + y \leq 0$

Original is True: For a given x take $y = 1 - x$.

ii) Negation $\forall x \in \mathbb{N}$, $\exists y \in \mathbb{Z}$ s.t. $x + y \leq 0$

Negation is True: For a given x take $y = -x$.

Original is False.

(2) (2.5 marks) Solve the equation $10x + 61 = 49 + 36x$ in $\mathbb{Z}_{59}$.

$36x - 10x = 61 - 49$ $\quad$ $26x = 12$

We need to find $(26)^{-1}$ in $\mathbb{Z}_{59}$. Euclidean algorithm:

$59 = 2 \cdot 26 + 7$

$26 = 3 \cdot 7 + 5$

$7 = 1 \cdot 5 + 2$

$5 = 2 \cdot 2 + 1$

$1 = 5 - 2 \cdot 2 = 5 - 2(7 - 1 \cdot 5) = -2 \cdot 7 + 3 \cdot 5 = -2 \cdot 7 + 3(26 - 3 \cdot 7) =$

$= 3 \cdot 26 - 11 \cdot 7 = 3 \cdot 26 - 11(59 - 2 \cdot 26) = -11 \cdot 59 + 25 \cdot 26$

$\Rightarrow (26)^{-1} = 25$ in $\mathbb{Z}_{59}$.

$x = (26)^{-1} \cdot 12 = 25 \cdot 12 = 300 \bmod 59 = (5 \cdot 59 + 5) \bmod 59$

$x = 5$

(3) (2.5 marks) Solve the recurrence relation

$$A(n) = 4(n-1)A(n-1) \text{ for } n \geq 2,\ A(1) = 2$$

$A(2) = 4(1)\cdot 2$, $A(3) = 4(2)\cdot 4(1)\cdot 2 = 4^2 \cdot 2! \cdot 2$

$A(4) = 4(3)\cdot 4(2)\cdot 4(1)\cdot 2 = 4^3 \cdot 3! \cdot 2$

Pattern guess: $A(n) = 4^{n-1}(n-1)!\cdot 2$

Proof by induction:

$A(1) = 4^{1-1}(0)!\cdot 2 = 2$

Assume $A(k) = 4^{k-1}(k-1)!\cdot 2$ Need to prove: $A(k+1) = 4^k \cdot k! \cdot 2$

$A(k+1) = 4(k+1-1)A(k) = 4(k)\cdot 4^{k-1}(k-1)!\cdot 2 = 4^k \cdot k! \cdot 2$

(4) (2.5 marks) Prove that for three sets A, B and C the equality

$$(A \cap B) \cup C = A \cap (B \cup C)$$

holds iff $C \subseteq A$.

a) If $C \subseteq A$ then $A \cup C = A$. We have

$LHS = (A \cap B) \cup C = (A \cup C) \cap (B \cup C) = A \cap (B \cup C) = RHS$

b) Now assume $LHS = RHS$.

Let $x \in C$. We want to show $x \in A$.

If $x \in A$ we are done. If not, $x \notin A \cap (B \cup C) = RHS = LHS$.

We have $x \notin (A \cap B) \cup C$ which contradicts $x \in C$. Thus $x \in A \Rightarrow$

$C \subseteq A$.

(2 marks)

(5) Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ where A and B are arbitrary sets.

We have to show that every element of the LHS is also an element of the RHS.

Let $X \in P(A)$. Then $X \subseteq A$ and $X \subseteq A \cup B \Rightarrow X \in P(A \cup B)$.

Let $X \in P(B)$. Then $X \subseteq B$ and $X \subseteq A \cup B \Rightarrow X \in P(A \cup B)$.

Thus LHS $\subseteq$ RHS.

(6) (2.5 marks) Let D be the relation defined on $\mathbb{Z}$ as follows: $\forall m, n \in \mathbb{Z}$

$$\{m\ D\ n\} \leftrightarrow 7|(m^3 - n^3)$$

Prove that this relation is an equivalence relation and describe the distinct equivalence classes of this relation.

Reflexive: $7 \mid (m^3 - m^3) \Rightarrow m D m$ ✓

Symmetry: $7 \mid (m^3 - n^3) \Rightarrow 7 \mid -(m^3 - n^3) = n^3 - m^3$, i.e. $m D n \Rightarrow n D m$.

Transitivity: Let $m D n$ and $n D p$. Then $7 \mid (m^3 - n^3)$ and $7 \mid (n^3 - p^3)$.

We have $7 \mid (m^3 - p^3) = (m^3 - n^3) + (n^3 - p^3)$.

Since $m^3 - n^3 = (m-n)(m^2 + mn + n^2)$ the equivalence classes of D are rougher (larger) then the equivalence classes of mod 7.

We have $0^3 = 0 \bmod 7$; $1^3 = 1 \bmod 7$; $2^3 = 1 \bmod 7$; $3^3 = 6 \bmod 7$; $4^3 = 1 \bmod 7$; $5^3 = 6 \bmod 7$; $6^3 = 6 \bmod 7$. The equivalence classes of D are $[0]$, $[1]$ and $[6]$.

(7) (2.5 marks) Let ρ be a binary relation on a set S. Than a binary relation called the inverse of ρ, denoted by ρ^{-1} is defined by $x\rho^{-1}y \leftrightarrow y\rho x$.

Now let (S, ρ) be a poset. Show that $(S, \rho^{-1}))$ is also a poset.

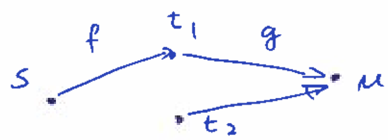
Reflexivity: $x\rho^{-1}x$ since $x\rho x$ (ρ is reflexive).

Antisymetry: Let $x\rho^{-1}y$ and $y\rho^{-1}x$. Then $y\rho x$ and $x\rho y$ and hence $x=y$ (ρ is antisymmetric).

Transitivity: Let $x\rho^{-1}y$ and $y\rho^{-1}z$. Then $y\rho x$ and $z\rho y$ $\Rightarrow$ $z\rho x$ (ρ is transitive) $\Rightarrow$ $x\rho^{-1}z$.

(8) (3 marks) Let $f : S \to T$ and $g : T \to U$ be functions. Prove or disprove the following assertions:
a) If $g \circ f$ is an injection, so is g.
b) If $g \circ f$ is a surjection, so is f.
c) If $g \circ f$ is a bijection, so are both f and g.

a) False. Consider the counterexample:



$g \circ f$ is injective but g is not.

b) False. Same counterexample as in a).
$g \circ f$ is surjective but f is not.

c) False. Same counterexample as in a).
$g \circ f$ is bijective but neither is f nor g.

(9) (2.5 marks) Determine the coefficient of x^3y^3 in the expansion of $(2x - y - 3)^8$.

$$\frac{8!}{3!\,3!\,2!}\,2^3(-1)^3(-3)^2 = (560) \times (8) \times (-1) \times 9 = -40320$$

(10) (2.5 marks) i) Under what conditions on n and m is the bipartite graph $K_{m,n}$ planar?

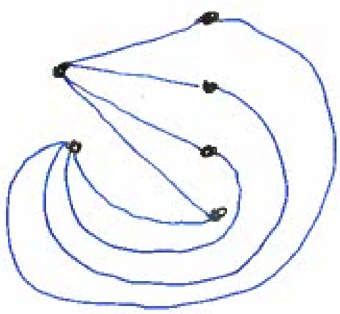
ii) Confirm Euler's formula for $K_{2,4}$. You might want to draw a picture to explain your conclusions.

i) $K_{1,n}$ is planar, so is $K_{m,1}$.

$K_{2,n}$ and $K_{m,2}$ are planar since we can always connect two vertices to any number of vertices with crossing arcs.

Any bipartite graph $K_{m,n}$ with $m \geq 3$ or $n \geq 3$ has $K_{3,3}$ as a subgraph and is not planar by Kuratowski's Theorem.

ii)



$n = 6 \quad a = 8 \quad r = 4$

$6 - 8 + 4 = 2$

(11) (2.5 marks) Let G be a simple graph. Prove that G is a tree iff G is connected and the addition of one arc to G results in a graph with exactly one cycle.

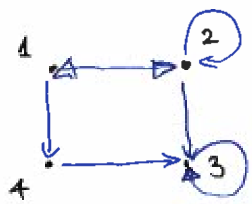
$\Rightarrow$ Let G be a tree. Then G is connected and there is precisely one path from any node to any other node. Connecting two nodes with a new arc will create precisely one cycle starting or ending at either of these nodes.

$\Leftarrow$ Let G be a connected graph whose one cycle was created by adding an arc. Removing this arc leaves a tree.

(12) (3 marks) Consider the binary relation $\rho = \{(1,2),(1,4),(2,1),(2,2),(2,3),(3,3),(4,3)\}$ on the set $S = \{1,2,3,4\}$.

a) Draw the associated directed graph and the adjacency matrix.

b) Determine the transitive closure of ρ by computing the reachability matrix (show details).



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$R = A \vee A^2 \vee A^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Transitive closure $= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,3),(4,3))$.

(13) (2.5 marks) Recall that K_n denotes the simple, complete graph with n vertices.

i) For what values of n does an Euler path exist in K_n?

ii) For what values of n does a Hamiltonian circuit exist in K_n?

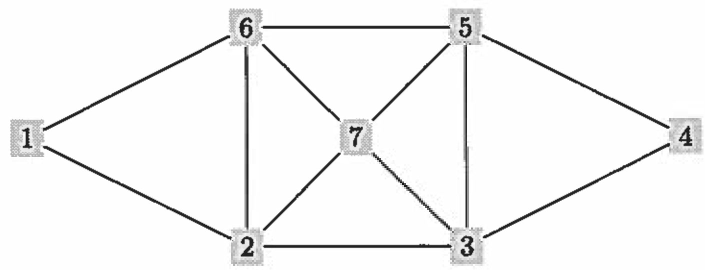
i) In k_n every node has a degree $n-1$.
If n is odd, Then every node in k_n has even degree. According to Euler's Theorem, Euler path exists.
If n is even, Then every node in k_n has odd degree. Since for $n \geq 4$, n even k_n has has more than 2 vertices of odd degree Euler path does not exist.
Conclusion: Euler path exists in k if n is odd or $n = 2$.

ii) Since in k_n every node is connected to every other node a Hamiltonian circuit can be constructed from the current node to any unvisited node. The exception is k_2.
Conclusion: Hamiltonian circuits exist in k_n, $n \geq 3$.

(14) (2.5 marks) For the graph below:

i) Write the nodes in a depth-first search starting at node 1 and following an alphabetical order.

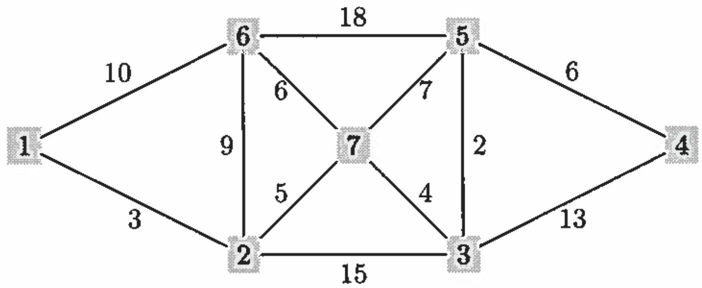
ii) Write the nodes in a breadth-first search starting at node 1 and following an alphabetical order.



i) $\{1, 2, 3, 4, 5, 6, 7\}$

ii) $\{1, 2, 6, 3, 7, 5, 4\}$

(15) (3 marks) For the weighted graph below, while describing every step in the algorithm you are using, find the shortest distance between node 1 and node 4.



Dijkstra's algorithm:

$D=\{1\}$

	1	2	3	4	5	6	7
d	0	3	∞	∞	∞	10	∞
p	-	1	1	1	1	1	1

; $D=\{1,2\}$

	1	2	3	4	5	6	7
d	0	3	18	∞	∞	10	8
p	-	1	2	1	1	1	2

$D=\{1,2,7\}$

	1	2	3	4	5	6	7
d	0	3	12	∞	15	10	8
p	-	1	7	1	7	1	2

; $D=\{1,2,7,6\}$

	1	2	3	4	5	6	7
d	0	3	12	∞	15	10	8
p	-	1	7	1	7	1	2

$D=\{1,2,7,6,3\}$

	1	2	3	4	5	6	7
d	0	3	12	25	14	10	8
p	-	1	7	3	3	1	2

; $D=\{1,2,7,6,3,5\}$

	1	2	3	4	5	6	7
d	0	3	12	20	14	10	8
p	-	1	7	5	3	1	2

The shortest path is of length 20 : $1 \to 2 \to 7 \to 3 \to 5 \to 4$

(16) (2 marks) For the graph from problem 15 describe step-by-step construction of a minimal spanning tree.