ProbStat - Clex 13 - Solutions

(1) a)

x \ y	0	5	10	15	p(x)
0	0.02	0.06	0.02	0.10	0.20
5	0.04	0.15	0.20	0.10	0.49
10	0.01	0.15	0.14	0.01	0.31
p(y)	0.07	0.36	0.36	0.21	1

b)

y	0	5	10	15
$p_{Y\mid X=10}$	$\frac{0.01}{0.31}$ = 0.032	$\frac{0.15}{0.31}$ = 0.484	$\frac{0.14}{0.31}$ = 0.452	$\frac{0.01}{0.31}$ = 0.032

c)

x	0	5	10
$p_{X\mid Y=5}$	$\frac{0.06}{0.36}$	$\frac{0.15}{0.36}$	$\frac{0.15}{0.36}$

$$E(X\mid Y=5) = 0\cdot\frac{0.06}{0.36} + 5\cdot\frac{0.15}{0.36} + 10\cdot\frac{0.15}{0.36} = 6.25$$

$$Var(X\mid Y=5) = 0^2\,\frac{0.06}{0.36} + 5^2\,\frac{0.15}{0.36} + 10^2\,\frac{0.15}{0.36} - (6.25)^2 = 13.02$$

d) $p_{XY}(0,0) = 0.02 \neq (0.07)(0.20) = p_X(0)p_Y(0)$. Dependent.

(2)

x \ y	0	1	2	p_X
0	0.12	0.05	0.03	0.20
2	0.05	0.35	0.10	0.50
5	0.03	0.10	0.17	0.30
p_Y	0.20	0.50	0.30	1

$$\mu_X = 0(0.20) + 2(0.50) + 5(0.30) = 2.5$$
$$\mu_Y = 0(0.20) + 1(0.50) + 2(0.30) = 1.1$$
$$\sigma_X^2 = 0^2(0.20) + 2^2(0.50) + 5^2(0.30) - (2.5)^2 = 3.25$$
$$\sigma_Y^2 = 0^2(0.20) + 1^2(0.50) + 2^2(0.30) - (1.1)^2 = 0.49$$

$$Cov(X,Y) = E(XY) - \mu_X\mu_Y = 2\cdot1\cdot(0.35) + 2\cdot2\cdot(0.1) + 5\cdot1\cdot(0.1) + 5\cdot2\cdot(0.17) - (2.5)(1.1)$$

$Cov(X,Y) = 0.55$, $\rho_{XY} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \dfrac{0.55}{\sqrt{(3.25)(0.49)}} = 0.4358$

(3) a) $p(6,2,4) = \dfrac{12!}{6!\,2!\,4!}(0.4)^6(0.1)^2(0.5)^4 = 0.035$

b) $p(x_A) = {}_{12}C_{x_A}\,(0.4)^{x_A}(0.6)^{12-x_A}$

c) $p(x_A, x_B) = \dfrac{12!}{x_A!\,x_B!\,(12-x_A-x_B)!}(0.4)^{x_A}(0.1)^{x_B}(0.5)^{12-x_A-x_B}$

d) $p(x_A, x_B\mid x_C=5) = \dfrac{12!/(x_A!\,x_B!\,5!)}{12!/(7!\,5!)}\;\dfrac{(0.4)^{x_A}(0.1)^{x_B}(0.5)^5}{(0.5)^7(0.5)^5}$

$$= \frac{7!}{x_A!\,x_B!}\;\frac{(0.4)^{x_A}(0.1)^{x_B}}{(0.5)^7}, \quad x_A + x_B = 7$$

(4) a) $p(x) = \displaystyle\int_0^{1-x} 24xy\,dy = 24x\cdot\frac{y^2}{2}\Big|_{y=0}^{y=1-x} = 12x(1-x)^2$

since the distribution is symmetric under $x \leftrightarrow y$:

$p(y) = 12y(1-y)^2$

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b) $$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{24xy}{12y(1-y)^2} = \frac{2x}{(1-y)^2} \qquad 0<x<1-y$$

By symmetry

$$p(y|x) = \frac{2y}{(1-x)^2}$$

c) $$p(x|Y=1/4) = \frac{2x}{(1-1/4)^2} = \frac{32x}{9}$$

$$E(X|Y=1/4) = \int_0^{3/4} x \cdot \frac{32x}{9}\,dx = \frac{32}{9}\cdot\frac{x^3}{3}\Big|_0^{3/4} = \frac{1}{2}$$

$$Var(X|Y=1/4) = \int_0^{3/4} x^2 \cdot \frac{32x}{9}\,dx - \left(\frac{1}{2}\right)^2 = \frac{32}{9}\cdot\frac{x^4}{4}\Big|_0^{3/4} - \frac{1}{4} = \frac{1}{32}$$

d) $p(x|y) \neq p(x)$. The variables are dependent.

5 $X_i \sim Binom(n, p_i)$, so $Var(X_i) = np_i(1-p_i)$

Let $I_t^{(i)}$ be the Bernolli RV which 1 if trial t has outcome in class i

Then $I_t^{(i)} \sim Bernoulli(p_i)$ and $X_i = \sum_{t=1}^{n} I_t^{(i)}$. We have

$$E(X_i, X_j) = E\left[\left(\sum_{t=1}^{n} I_t^{(i)}\right)\left(\sum_{s=1}^{n} I_t^{(j)}\right)\right] = \sum_{t=s} E\left(I_t^{(i)} E_s^{(j)}\right) +$$

$$+ \sum_{t\neq s} E\left(I_t^{(i)} E_s^{(j)}\right) = 0 + \sum_{t\neq s} E\left(I_t^{(i)}\right)E\left(I_s^{(j)}\right) = \sum_{t\neq s} p_i p_j =$$

$$= (n^2-n)\,p_i p_j$$

The expected value above factorizes since the trials are independent.

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j) = (n^2-n)p_ip_j - np_i np_j$$

$$= -np_ip_j$$

$$Corr(X_i, X_j) = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i)Var(X_j)}} = \frac{-np_ip_j}{\sqrt{np_i(1-p_i)\,np_j(1-p_j)}}$$

$$Corr(X_i, X_j) = \frac{-np_ip_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}}$$