Probstat - Clex 6 - Solutions

1) a) For $0 < x < 1$, $F(x) = \int_0^x 90\, u^8(1-u)\, du = 90\left(\frac{u^9}{9} - \frac{u^{10}}{10}\right)\Big|_{u=0}^{u=x} =$

$= 10x^9 - 9x^{10}$

$$F(x) = \begin{cases} 0 & x \le 0 \\ 10x^9 - 9x^{10} & 0 < x < 1 \\ 1 & 1 \le x \end{cases}$$

b) $P(0.25 < X < 0.5) = F(0.5) - F(0.25) = [10\cdot(0.5)^9 - 9(0.5)^{10}] -$

$[10(0.25)^9 - 9(0.25)^{10}] = 0.0107$

c) $E(X) = \int_0^1 x \cdot 90x^8(1-x)\, dx = 90\int_0^1 (x^9 - x^{10})\, dx = \left(9x^{10} - \frac{90}{11}x^{11}\right)\Big|_0^1 =$

$= 9 - \frac{90}{11} = \frac{9}{11} = 0.8182$

$Var(X) = \int_0^1 x^2 \cdot 90x^8(1-x)\, dx - \left(\frac{9}{11}\right)^2 = \left(\frac{90}{11}x^{11} - \frac{90}{12}x^{12}\right)\Big|_0^1 - \frac{81}{121} =$

$= \frac{15}{22} - \frac{81}{121} = \frac{3}{242} \approx 0.0124$

2) $\int_{-\infty}^{\infty} c e^{-x-e^{-x}} = \{u = e^{-e^{-x}},\ du = e^{-x} e^{-e^{-x}} dx = e^{-x-e^{-x}} dx\}$

$= c\int_0^1 du = cu\Big|_0^1 = c = 1$

$F(x) = \int_{-\infty}^x e^{-t-e^{-t}}\, dt = \{u = e^{-e^{-t}}\} = \int_0^{e^{-e^{-x}}} du = e^{-e^{-x}}$

3) $E(Area) = E(\pi r^2) = \int_9^{11} \pi r^2 \cdot \frac{3}{4}[1 - (10-r)^2]\, dr = \frac{3\pi}{4}\int_9^{11}(-99r^2 + 20r^3 - r^4)\, dr$

$= \frac{3\pi}{4}\left(-33r^3 + 5r^4 - r^5/5\right)\Big|_9^{11} = 314.79$

4) a) $\int_0^1 k(x^2 + x)\, dx = k\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Big|_0^1 = k \cdot \frac{5}{6} = 1 \implies k = \frac{6}{5}$

b) For $0 < x < 1$, $F(x) = \int_0^x \frac{6}{5}(u^2 + u)\, du = \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)$

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right) & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

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c) $P(X < 1/2) = F(1/2) = \frac{6}{5}\left(\frac{1}{24} + \frac{1}{8}\right) = \frac{1}{5}$

Now binomial: $p = 1/5$, $n = 38$

$P(Y=8) + P(Y=9) = {}_{38}C_8 (1/5)^8 (4/5)^{30} + {}_{38}C_9 (1/5)^9 (4/5)^{29} =$

$= 0.1550 + 0.1291 = 0.2841$

(5) $X \sim \text{Uniform}(1.5, 11.5)$; $a = 1.5$, $b = 11.5$

a) $E(X) = \frac{a+b}{2} = \frac{1.5 + 11.5}{2} = 6.5$ min

$\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(11.5 - 1.5)^2}{12} = \frac{100}{12} \approx 8.33 \text{ min}^2$

b) $F(x) = \begin{cases} 0 \\ \frac{1}{10}(x - 1.5) \\ 1 \end{cases}$, $P(X < 2) = F(2) = \frac{1}{10}(2 - 1.5) = 0.05$

c) $P(5 < X < 10) = F(10) - F(5) = \frac{1}{10}(10 - 1.5) - \frac{1}{10}(5 - 1.5) = 0.5$

d) $F(a) = 0.99$, $\frac{1}{10}(a - 1.5) = 0.99 \Rightarrow a = 11.4$ min

e) $P(X > 7) = 1 - F(7) = 1 - \frac{1}{10}(7 - 1.5) = 0.45$

Now binomial, $p = 0.45$, $n = 10$

$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1) = 1 - {}_{10}C_0 (0.45)^0 (0.55)^{10} - {}_{10}C_1 (0.45)^1 (0.55)^9 =$

$= 0.9768$

(6) $X \sim \text{Exp}(0.1)$ since $\lambda = \frac{6 \text{ calls}}{60 \text{ min}} = 0.1 \frac{\text{call}}{\text{min}}$, $F(x) = 1 - e^{-\lambda x}$

a) $P(X > 10) = 1 - F(10) = 1 - \left(1 - e^{-0.1(10)}\right) = e^{-1} \approx 0.3678$

b) $P(X < 30) = F(30) = 1 - e^{-0.1(30)} = 0.9502$

c) $P(X < 20 + 8 \mid X > 20) = P(X < 8) = F(8) = 1 - e^{-0.1(8)} = 0.5507$

d) $F(a) = 0.99$, $1 - e^{-0.1(a)} = 0.99$, $e^{-0.1a} = 0.01$

$a = \frac{\ln(0.01)}{-0.1} = 46.05$ min

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7 $\lambda = \frac{1}{5}$ cracks/mile

a) $X \sim \text{Poisson}\left(\frac{1}{5} \cdot 10\right)$

$$P(X=2) + P(X=3) = \frac{e^{-2} \cdot 2^2}{2!} + \frac{e^{-2} \cdot 2^3}{3!} = 0.2707 + 0.1804 = 0.4511$$

b) $Y \sim \exp(1/5)$

$$P(Y > 10) = 1 - F(10) = 1 - (1 - e^{-1/5 \cdot 10}) = e^{-2} = 0.1353$$

c) $P(12 < Y < 15) = F(15) - F(12) = (1 - e^{-1/5 \cdot 15}) - (1 - e^{-1/5 \cdot 12}) = 0.0409$

d) Memoryless:

$$P(Y > 5 + 10 \mid Y > 5) = P(Y > 10) = 0.1353$$