Probstat - Clex 8 - Solutions

1) $y = \sqrt{x}$ is a monotone increasing function on $[0, \infty)$.

$$p(y) = p(x(y))\,\frac{dx}{dy} = \frac{1}{(1+y^2)^2}\cdot 2y = \frac{2y}{(1+y^2)^2}, \quad y \geq 0$$

For $y \geq 0$ $$F(y) = \int_0^y p(t)\,dt = \int_0^y \frac{2t}{(1+y^2)^2}\,dt = -\frac{1}{1+t^2}\Big|_0^y = 1 - \frac{1}{1+y^2}$$

$$F(y) = \begin{cases} 0 & y < 0 \\ 1 - \frac{1}{1+y^2} & y \geq 0 \end{cases}$$

2) The waiting time is $X \sim \text{Erlang}(4.7, 40)$

a) $$p(x) = \frac{4.7^{40}\cdot x^{39}\cdot e^{-4.7x}}{39!}$$

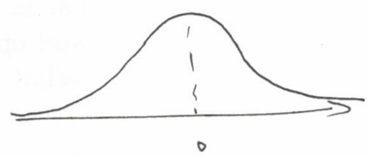
b) $E(X) = \frac{r}{\lambda} = \frac{40}{4.7} = 8.51$, $\quad Var(X) = \frac{r}{\lambda^2} = \frac{40}{(4.7)^2} = 1.81$

c) $$p(X > 10) = 1 - F(10) = 1 - \left[1 - \sum_{k=0}^{39} \frac{e^{-4.7(10)}(4.7\times 10)^k}{k!}\right] = 0.1356$$

3) a) $p(X > 1.22) = 0.1171$

b) $p(-0.78 < X < 2.04) = 0.7522$

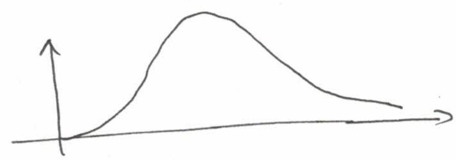
c) $P_{95} \rightarrow X = 1.71$

d) 

4) a) $p(X < 4.23) = 0.9762$

b) $p(2.8 < X < 3.4) = 0.1436$

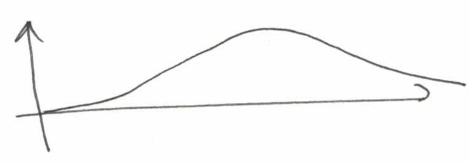
c) $P_{95} \rightarrow X = 3.82$

d) 

5) a) $p(X > 7.23) = 0.7802$

b) $p(8.0 < X < 12.4) = 0.3790$

c) $P_{95} \rightarrow X = 19.6751$

d) 

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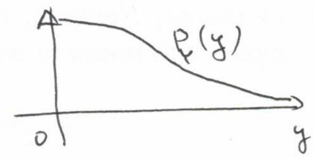
6) $F_Y(y) = p(Y \le y) = p(|X| \le y) = p(-y \le X \le y) = 2p(0 \le X \le y)$

$$= 2\int_0^y \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

$$p_Y(y) = F'_Y(y) = \frac{2}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}, \quad y > 0$$



$$E(Y) = \int_{-\infty}^{\infty} y\, p_Y(y)\,dy = \int_0^{\infty} \frac{2}{\sqrt{2\pi}\,\sigma}\, y\, e^{-\frac{y^2}{2\sigma^2}}\,dy \quad \left\{ u = \frac{y^2}{2\sigma^2},\ du = \frac{y}{\sigma^2}\,dy \right\}$$

$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi}\,\sigma} \cdot \sigma^2 e^{-u}\,du = \sqrt{\frac{2}{\pi}}\,\sigma \left(-e^{-u}\right)\Big|_0^{\infty} = \sigma\sqrt{\frac{2}{\pi}}$$

7) $E(Y) = \int_0^1 x^2 \cdot 4x(1-x^2)\,dx = 4\int_0^1 (x^3 - x^5)\,dx = 4\left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Big|_0^1 = \frac{1}{3}$

On $[0,1]$, $y = x^2$ is an invertible function. We can use the change of variables formula for the density.

$$p_Y(y) = p_X(x(y)) \cdot \frac{dx}{dy} = 4\left(y^{1/2} - y^{3/2}\right) \cdot \frac{1}{2\sqrt{y}} = 2(1-y)$$

$$E(Y) = \int_0^1 y\, p_Y(y)\,dy = \int_0^1 y \cdot 2(1-y)\,dy = 2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$