§1.Sample Spaces and Axioms of Probability

Probability theory ⇒ the language, myriad of techniques, and algorithms used for analysing and predicting uncertain events.

Uncertainties are epistemic if the modeller can reduce them by refining the models, or by gathering more data. They are aleatory if the modeller does not see a possibility to reduce them.

History

- 1564: Cardano ⇒ Liber de ludo aleae ("Book on Games of Chance")
- Mid 17th Century: Pascal and Fermat ⇒ gambling games
- 1930's: Kolmogorov ⇒ modern rigorous mathematical foundation

Figure 1: Pascal; Fermat

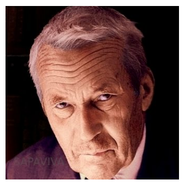
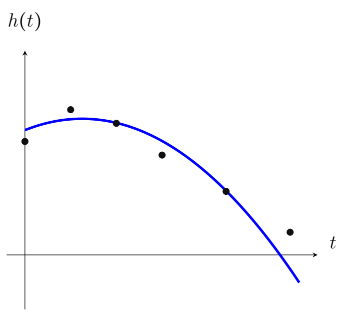


Figure 2: Kolmogorov

Modelling

Modern models in the sciences and engineering typically involve both deterministic and stochastic (random) terms.

> **Example 1**
>
> Projectile motion.
>
> $$h(t) = h_0 + vt - \frac{1}{2}gt^2 + \varepsilon(t)$$



Probability provides the basis for the theory and practice of statistical inference and analysis of data: hypothesis testing and confidence intervals.

Discrete Sample Spaces

Random experiments $\Rightarrow$ experiments which when repeated under identical circumstances yet produce different outcomes.

Definition 1

The **sample space** of a random experiment is the set of all possible outcomes of the experiment. The possible outcomes are called **sample points**.

Example 2

i Toss a coin: $S = \{H, T\}$

ii Toss a die: $S = \{1, 2, 3, 4, 5, 6\}$ or $S = \{even, odd\}$

iii Select six numbers at random from 49: $S = \{(1, 2, 3, \ldots, 6) \ldots, (44, 45, \ldots, 49)\}$

iv Number of attempts at a CFA exam before success:

$S = \{1, 2, 3 \ldots\}$ - *countably infinite (but discrete)*

v Weight of a newborn fawn of a white tailed deer:

$S = [2, 4]\,\text{kg}$ - *uncountably infinite*

Axioms of Probability

Definition 2

An **event** is a subset of a sample space.

Example 3

Toss two dice: $S = \{(1,1), (1,2), \ldots, (2,1), (2,2), \ldots, (6,6)\}$
Event A: "Sum is 10" $A = \{(4,6), (5,5), (6,4)\}$

Remark
Indistinguishable dice exist in quantum mechanics

Example 4

Weight of fawns: $S = [2, 4]\,\text{kg}$
Event B = $[3.5, 4]$ $\Rightarrow$ very heavy newborn fawns.

Definition 3

We say an event occurs if one of its sample points contained in the event occurs.

One way to define probabilities is via **relative frequencies**. A random experiment is repeated $n-$times. Event A occurs $n(A)$ times. The relative frequency of A is $\frac{n(A)}{n}$

Example 5

Toss a coin 5000 times. Let T be the event that tails is observed. Suppose that $n(T) = 3000$. Then

$$\frac{n(T)}{n} = \frac{3000}{5000} = 0.6$$

Definition 4

If A is an event the **probability** (relative frequency interpretation) of A is

$$\mathbb{P}(A) = \lim_{n\to\infty} \frac{n(A)}{n}$$

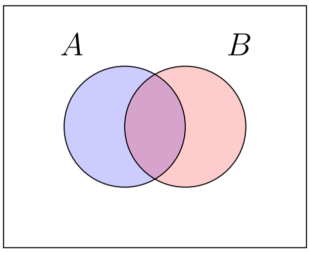
Remark

i We assume that the limit exists.

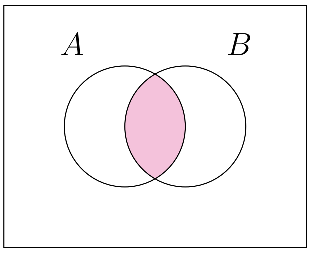
ii Repeating random experiments infinitely many times is not possible.

iii Bayesian's have a different approach.

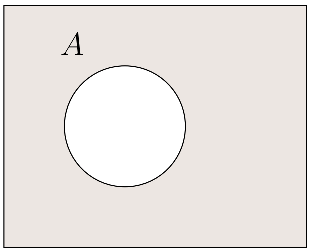
Boolean Algebra Review



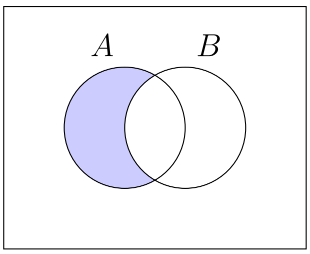
(a) $A \cup B$; (or)



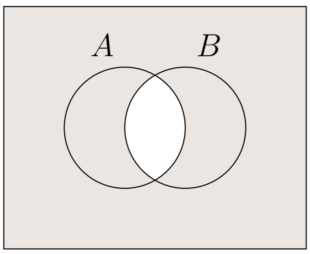
(b) $A \cap B$; (and)



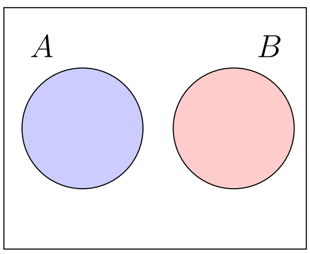
(c) $\overline{A} = A'$, not A



(d) $A \cap \overline{B}$



(e) $\overline{A} \cup \overline{B}$



(f) $A \cap B = \varnothing$; (disjoint)

Axioms of Probability

1. $P(A) \geq 0$
2. $P(S) = 1$
3. If $A_1, A_2, \ldots$ are mutually exclusive, $A_i \cap A_j = \varnothing \quad \forall i \neq j$ then

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} P(A_i)$$

Remark
These axioms reflect the common intuitions about relative frequencies.