§12. Poisson Random Variables

Definition 1

A random variable, X, is said to be a **Poisson random variable**, $X \sim \text{Poisson}(\lambda)$, if it has pmf

$$P(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots, \qquad \lambda > 0$$

Remark

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}\cdot e^{\lambda} = 1$$

Proposition

The expected value and variance are

$$E(X) = \lambda \qquad \text{Var(X)} = \lambda$$

Proof.

$$E(X) = \sum_{x=0}^{\infty} xp(x) = \sum_{x=1}^{\infty} xe^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} e^{-\lambda}\frac{\lambda\cdot\lambda^x}{(x-1)!} = \;= \lambda\sum_{x=1}^{\infty}\frac{e^{-x}\lambda^{x-1}}{(x-1)!} = \lambda\cdot 1 = \lambda$$

□

Remark

The Poisson RV does not primarily describe a particular type of experiment. Rather the Poisson distribution has been empirically observed in many different applications. Typical cases are when we count the number of occurrences of the same unpredictable event over time.

Example 1

There are 5.9 hurricanes on average every year. What is probability that in a given year there are more than 6 hurricanes?

Solution

$$\begin{aligned}
P(X > 6) &= 1 - P(x \le 6) \\
&= 1 - [P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6)] \\
&= 1 - \left[\frac{e^{-5.9}(5.9)^0}{0!} + \frac{e^{-5.9}(5.9)^1}{1!} + \frac{e^{-5.9}(5.9)^2}{2!} + \frac{e^{-5.9}(5.9)^3}{3!}\right. \\
&\qquad \left. + \frac{e^{-5.9}(5.9)^4}{4!} + \frac{e^{-5.9}(5.9)^5}{5!} + \frac{e^{-5.9}(5.9)^6}{6!}\right] \\
&= 1 - [0.0027 + 0.6162 + 0.0477 + 0.0938 + 0.1383 + 0.1632 \\
&\qquad +0.1605] \\
&= 0.3776
\end{aligned}$$

Example 2

There are an average of 12 snowy days in Mont-Tremblant in February. What is the probability that there will only be 4 snowy days in the first two weeks of February?

$$\lambda = \frac{12}{2} = 6 \qquad P(4) = e^{-6}\frac{6^4}{4!} = 0.1339$$

The Poisson random variable is a limit of binomial random variable when the binomial events have low probability of success $p \to 0$ but $np \to \lambda =$ constant

Indeed

$$P_{\text{binom}}(x) = {}_nC_x p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!}p^x(1-p)^{n-x}$$

$$P_{\text{binom}}(x) = \frac{p}{1-p}\cdot\frac{n-x+1}{x}P_{\text{binom}}(x-1) = \frac{np - p(x-1)}{(1-p)x}P_{\text{binom}}(x-1)$$

Taking the limit $p \to 0, \quad np \to \lambda$

$$P(x) = \frac{\lambda}{x}P(x-1) \qquad x = 1, 2, \ldots$$

$$P(1) = \frac{\lambda}{1}P(0), \quad P(2) = \frac{\lambda}{2}P(1) = \frac{\lambda^2}{2\cdot 1}P(0), \quad \ldots, \quad P(x) = \frac{\lambda^x}{x!}P(0)$$

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty}\frac{\lambda^x}{x!}P(0) = P(0)\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = P(0)e^{\lambda} = 1$$

$$\Rightarrow P(0) = e^{-\lambda} \quad \text{and} \quad P(x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

Example 3

A Real Analysis textbook has 760 pages. The probability for a typo is $p = 0.006$ per page. What is the probability that there are 5 typos in the book (one typo per page).

$$P_{\text{binom}}(5) = {}_{760}C_5(0.006)^5(0.994)^{755} = 0.17244$$

Poisson approximation: $\lambda = np = 760(0.006) = 4.56$

$$P_{\text{poisson}}(5) = e^{-4.56}\frac{(4.56)^5}{5!} = 0.17190$$