§13. Poisson Process

The Poisson counting process is a collection of random variables $\{N(t), t \geq 0\}$ such that:

i. $N(0) = 0 \quad ; \quad N(t) \in \mathbb{N}$

ii. Has independent increments, i.e. $N(b) - N(a)$ and $N(d) - N(c)$ are independent random variables for disjoint intervals $[a, b] \cap [c, d] = \varnothing$.

iii. The number of events in any interval of length, t, is a Poisson random variable with parameter λt :

$$N(t) \sim \text{Poisson}(\lambda t)$$

Example 1

Customers walk into a grocery store at a rate of 4 per minute:

a. Find the probability of exactly 6 customers arrive in an interval of 2 minutes

b. Determine the probability of at least 3 arrive in 3 minutes.

Solution

a. $\lambda t = 8$

$$P(6) = \frac{e^{-8} 8^6}{6!} = 0.122138$$

b. $\lambda t = 12$

$$\begin{aligned} P(x \geq 3) &= 1 - P(0) - P(1) - P(2) \\ &= 1 - \left[\frac{e^{-12} \cdot 12^\circ}{0!} + \frac{e^{-12} \cdot 12^1}{1!} + \frac{e^{-12} \cdot 12^2}{2!} \right] = 0.999478 \end{aligned}$$

Example 2

Consider a Poisson process with a rate of $\lambda = 3$ per hour. Let x_1 be the number of events in the first hour and x_2 be the number of events in the second hour. Let's compute the probability that $x_1 + x_2 = 7$. We have

$$
\begin{aligned}
P\left(x_1 + x_2 = 7\right) &= \sum_{x_1=0}^{7} P\left(X_1 = x_1\right) \cdot P\left(X_2 = 7 - x_1\right) \\
&= \sum_{x_1=0}^{7} \frac{e^{-3} \cdot 3^{x_1}}{x_1!} \cdot \frac{e^{-3} \cdot 3^{7-x_1}}{(7-x_1)!} \\
&= \frac{e^{-6}}{7!} \sum_{x_1=0}^{7} \frac{7!}{(x_1)!\,(7-x_1)!} 3^{x_1} 3^{7-x_1} \\
&= \frac{e^{-6}}{7!} \sum_{x_1=0}^{7} {}_7C_x\, 3^{x_1} 3^{7-x_1} \\
&= \frac{e^{-6}}{7!} (3+3)^7 \\
&= \frac{e^{-6} \cdot 6^7}{6!}
\end{aligned}
$$

But this should have been obvious since $\lambda t = 3 \cdot 2 = 6$ for the combined two hours.