§14. Continuous Random Variables

A random variable, X, which takes values on an interval $\subseteq \mathbb{R}$ is a continuous random variable.

Example 1

Temperature at the airport at 6 pm on Feb 25^{th}. What is the probability that this temperature will be $3.4752\,°C$?

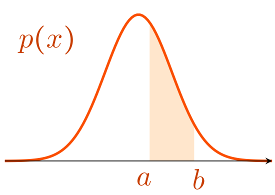
Two types of possible answers:

- $P(X = 3.4752) = 0$ Since the real numbers are uncountable, there is no way to assign probability to any specific value; so we assign zero probability.
- This is a non-physical (nonsensical question). There is no way to measure temperature with infinite precision.
- However we can measure the temperature with some precision say $3.4752 \pm 0.0001\,°C$. So we can claim the temperature is within the interval $[3.4751, 3.4753]\,°C$. Continuous physical quantities are described with intervals and so we will assign probabilities to intervals.

Definition 1

A continuous random variable is specified via its **probability density function** (pdf), $p(x)$. The pdf satisfies:

1. $p(x) \geq 0$
2. $\int_{-\infty}^{\infty} p(x)\,dx = 1$
3. $P(a \leq X \leq b) = \int_a^b p(x)\,dx$



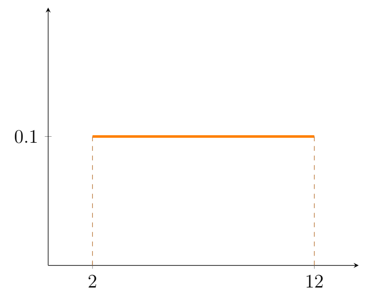
probability = area

Remark
The pdf does not have the meaning of probability. Its anti-derivative does!

Example: Uniform Distribution on $[2, 12]$

$$p(x) = \begin{cases} 0.1 & ; \quad 2 \le x \le 12 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} p(x) = \int_{2}^{12} 0.1\,dx = 0.1x]_2^{12} = 1$$



Example: Exponential Density

$$p(x) = \begin{cases} e^{-x} & ; \quad x \ge 0 \\ 0 & ; \quad x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} p(x) = \int_{0}^{\infty} e^{-x}\,dx = -e^{-x}]_0^{\infty} = 1$$

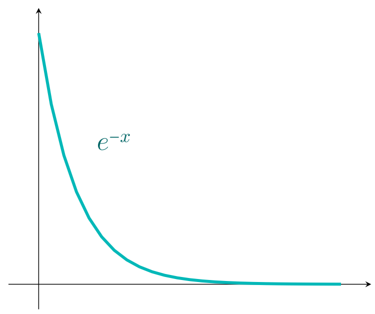


Figure 1: Exponential Distribution

Example: Cauchy Density

$$p(x) = \frac{1}{\pi(1+x^2)}$$

$$\begin{aligned} \int_{-\infty}^{\infty} p(x) = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx &= \frac{1}{\pi}\arctan(x)\Big]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] \\ &= 1 \end{aligned}$$

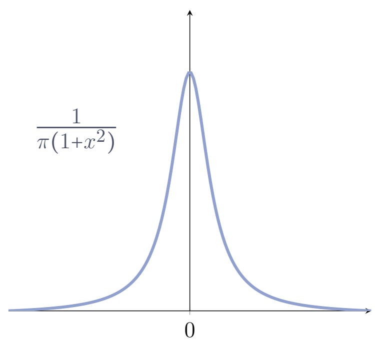


Figure 2: Cauchy Distribution

Example 2

What value of the constant k will make the $p(x)$ below a pdf?

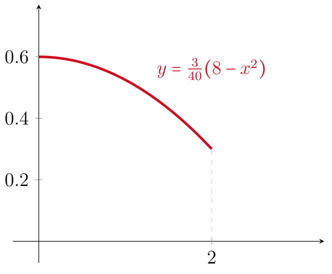
$$p(x) = \begin{cases} k(8 - x^2) & ; \quad 0 \le x \le 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Solution

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^2 k(8 - x^2)\,dx$$

$$1 = k\left[8x - \frac{x^3}{3}\right]_0^2$$

$$1 = k\left(16 - \frac{8}{3}\right) \quad \Rightarrow \quad 1 = \frac{40}{3}k \quad \Rightarrow \quad k = \frac{3}{40}$$



Definition 2

The **expected value** of a random variable X with pdf $p(x)$ is

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot p(x)\,dx$$

The **variance** of X is

$$\sigma^2 = \text{Var}(X) = E(X - E(X))^2 = \int_{-\infty}^{\infty} (X - \mu)^2 \cdot p(x)\,dx$$

Proposition

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

Proof.

$$\begin{aligned} \text{Var}(X) &= \int_{-\infty}^{\infty} (x - \mu)^2 \cdot p(x)\,dx \\ &= \int_{-\infty}^{\infty} x^2 \cdot p(x)\,dx - 2\mu \int_{-\infty}^{\infty} x \cdot p(x)\,dx + \mu^2 \int_{-\infty}^{\infty} p(x)\,dx \\ &= \int_{-\infty}^{\infty} x^2 \cdot p(x)\,dx - 2\mu \cdot \mu + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

□

Example 3

Determine $E(X)$ and $\sigma(X)$ for

$$p(x) = \begin{cases} e^{-x} & ; \quad x \geq 0 \\ 0 & ; \quad x < 0 \end{cases}$$

Solution

$$E(X) = \int_0^{\infty} x \cdot e^{-x}\, dx = [-xe^{-x}]_0^{\infty} + \int_0^{\infty} e^{-x}\, dx = [-e^{-x}]_0^{\infty} = 1$$

$$\begin{aligned} \text{Var}(X) = \int_0^{\infty} (x^2 \cdot e^{-x})\, dx - (1)^2 &= \left[-x^2 e^{-x}\right]_0^{\infty} + \int_0^{\infty} 2xe^{-x}\, dx - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

Example 4

Determine $E(X)$ and $\sigma(X)$ for the Cauchy distribution

$$p(x) = \frac{1}{\pi(1 + x^2)}$$

Solution

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi(1 + x^2)}\, dx \stackrel{?}{=} 0$$ Not really. This is $\infty - \infty$

$$\text{Var}(X) = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\pi(1 + x^2)}\, dx - (0)^2 = \infty$$

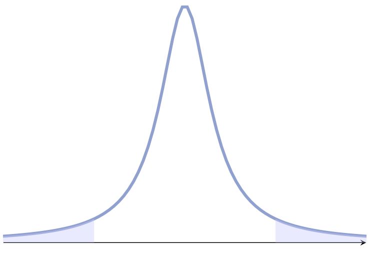
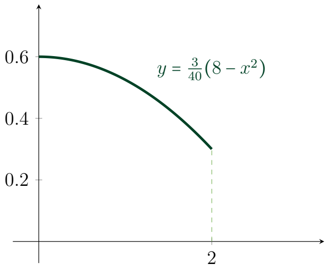


Figure 3: thick tails

Example 5

Determine $E(X)$ and $\sigma(X)$ for the distribution

$$p(x) = \begin{cases} \frac{3}{40}(8 - x^2) & ; \quad 0 \leq x \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$



Solution

$$E(X) = \int_0^2 x \cdot \frac{3}{40}(8 - x^2)\, dx = \frac{3}{40}\left(4x^2 - \frac{x^4}{4}\right)\Big|_0^2 = \frac{3}{40}(16 - 4) = \frac{9}{10}$$

$$\begin{aligned}\text{Var}(X) &= \int_0^2 x^2 \cdot \frac{3}{40}(8 - x^2)\, dx - \left(\frac{9}{10}\right)^2 = \frac{3}{40}\left(\frac{8x^3}{3} - \frac{x^5}{5}\right)\Big|_0^2 - \frac{81}{100} \\ &= \frac{31}{100}\end{aligned}$$

$$\sigma(X) = \frac{\sqrt{31}}{10}$$

Definition 3

The **cumulative probability function** (cdf) of a random variable with pdf $p(x)$ is

$$F(X) = P(X \le x) = \int_0^\infty p(t)\, dt$$

Note: $F(-\infty) = 0, F(\infty) = 1$ and $F(X)$ is never decreasing.

Example 6

Consider the random variable, X, with pdf

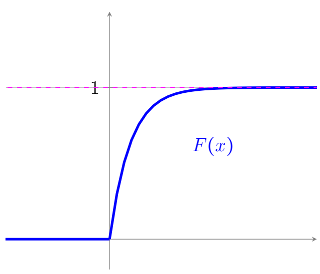
$$p(x) = \begin{cases} e^{-x} & ; \quad x \ge 0 \\ 0 & ; \quad x < 0 \end{cases}$$

For $x < 0$, $F(x) = 0$.

For $x \ge 0$, $F(x) = \int_0^x e^{-t}\, dt = -e^{-t}\big|_0^x = 1 - e^{-x}$

Thus,

$$F(x) = \begin{cases} 1 - e^{-x} & ; \quad x \ge 0 \\ 0 \quad ; & \quad x < 0 \end{cases}$$

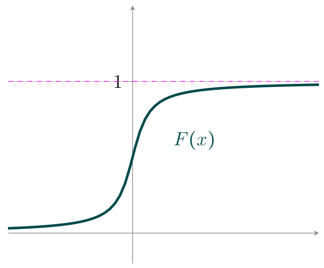


Example 7

Cumulative probability function for Cauchy density

$$p(x) = \frac{1}{\pi(1 + x^2)}$$

$$
\begin{aligned}
F(x) = \int_{-\infty}^{x} \frac{1}{\pi} \cdot \frac{1}{1+t^2}\, dt &= \frac{1}{\pi} \cdot \arctan(t)\Big|_{-\infty}^{x} \\
&= \frac{1}{\pi}\left(\arctan(x) + \frac{\pi}{2}\right) \\
&= \frac{1}{\pi}\arctan(x) + \frac{1}{2}
\end{aligned}
$$



Example 8

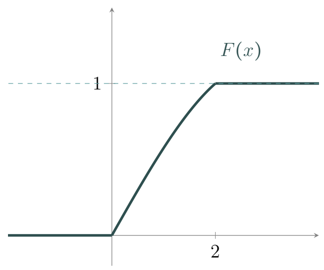
Consider the random variable, X, with pdf

$$
p(x) = \begin{cases} \frac{3}{40}(8 - x^2) & ; \quad 0 \le x \le 2 \\ 0 & ; \quad \text{otherwise} \end{cases}
$$

For $0 \le x \le 2$

$$
F(x) = \int_0^x \frac{3}{40}(8 - t^2)\, dt = \frac{3}{40}\left(8t - \frac{t^3}{3}\right)\Big|_0^x = \frac{3}{40}\left(8x - \frac{x^3}{3}\right) = \frac{3x}{5} - \frac{x^3}{40}
$$

$$
F(x) = \begin{cases} 0 & ; \quad x < 0 \\ \frac{3x}{5} - \frac{x^3}{40} & ; \quad 0 \le x \le 2 \\ 1 & ; \quad x > 2 \end{cases}
$$



Continuous Uniform Distribution on $[a, b]$

$$
p(x) = \begin{cases} \frac{1}{b-a} & ; \quad a \le X \le b \\ 0 & ; \quad \text{otherwise} \end{cases}
$$

$$
E(X) = \int_a^b \frac{x}{b-a}\, dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}
$$

$$
\text{Var}(X) = \int_a^b \frac{x^2}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2 = \frac{x^3}{3(b-a)}\Big|_a^b - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}
$$

$$
F(X) = \int_a^x \frac{1}{b-a}\, dx = \frac{x-a}{b-a} \quad \text{for} \quad a \le x \le b
$$

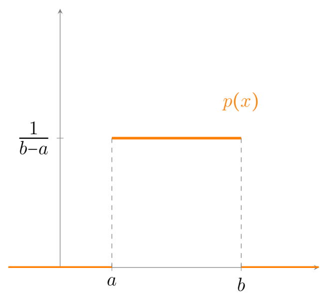


Figure 4: Continuous Uniform Distribution

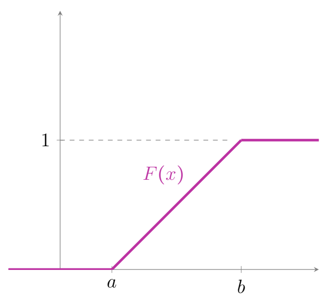


Figure 5: Cumulative Distribution

Example 9

X is uniform on $1 \le X \le 6$. Determine $P(X \ge 4 \,|\, X \ge 2)$.

Solution

$$P(X \ge 4 \,|\, X \ge 2) = \frac{P(X \ge 4)}{P(X \ge 2)} = \frac{1 - F(4)}{1 - F(2)} = \frac{1 - \frac{4-1}{6-1}}{1 - \frac{2-1}{6-1}} = \frac{2/5}{4/5} = \frac{1}{2}$$