§15. Exponential Random Variables

Definition 1

A **exponential random variable** X has pdf:

$$p(x) = \begin{cases} \lambda e^{-\lambda(x-a)} & ; \quad x \geq a \quad , \quad \lambda > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$X \sim \exp(\lambda, a)$

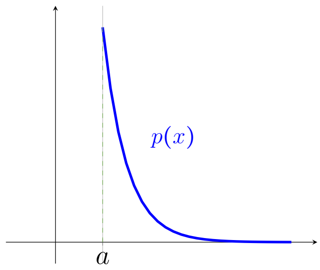

Figure 1: Graph of Exponential Distribution

Let's compute the expected value, variance, and cumulative distribution.

$$\begin{aligned} E(X) = \int_a^\infty x \cdot \lambda e^{-\lambda(x-a)}\, dx &= x \cdot \lambda \cdot \frac{e^{-\lambda(x-a)}}{-\lambda}\Bigg|_a^\infty - \int_a^\infty \lambda \cdot \frac{e^{-\lambda(x-a)}}{-\lambda}\, dx \\ &= a + \frac{e^{-\lambda(x-a)}}{-\lambda}\Bigg|_a^\infty \\ &= a + \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} \operatorname{Var}(X) = \int_a^\infty x^2 \cdot \lambda e^{-\lambda(x-a)}\, dx - \left(a + \frac{1}{\lambda}\right)^2 &= \left(a^2 + \frac{2a}{\lambda} + \frac{2}{\lambda^2}\right) - \left(a + \frac{1}{\lambda}\right)^2 \\ &= \frac{1}{\lambda^2} \end{aligned}$$

$$\sigma(X) = \frac{1}{\lambda}$$

$$F(x) = \int_a^x \lambda e^{-\lambda(x-a)}\, dt = \lambda \cdot \frac{e^{-\lambda(x-a)}}{-\lambda}\Bigg|_a^x = 1 - e^{-\lambda(x-a)} \quad , \quad x \geq a$$

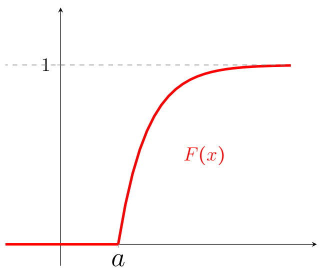

Figure 2: Graph of Cumulative Distribution

Example 1

The waiting time for the secretary of a family physician to pick up the phone from the automatic system is exponential $X \sim \exp\left(\frac{1}{4}, 0\right)$ where time is in minutes.

a. What is the expected waiting time before the secretary picks the phone and what is the standard deviation?

b. What is be probability that the patient has to wait mare than 10 minutes ?

Solution

a. $E(X) = a + \dfrac{1}{\lambda} = 0 + \dfrac{1}{1/4} = 4$ min, $\qquad \sigma(x) = \dfrac{1}{\lambda} = \dfrac{1}{1/4} = 4$ min

b. $P(X > 10) = 1 - F(10) = 1 - \left(1 - e^{-\frac{1}{4}(10-0)}\right) = e^{-2.5} = 0.0821$

Note that: If $X \sim \exp(\lambda, a)$ then

$$P(X \geq s + t \,|\, X \geq s) = \frac{1 - \left(1 - e^{s+t-a}\right)}{1 - \left(1 - e^{-\lambda(s-a)}\right)} = \frac{e^{-\lambda(s+t-a)}}{e^{-\lambda(s-a)}} = e^{-\lambda t}$$

$$P(X \geq t + a) = 1 - \left(1 - e^{-\lambda(t+a-a)}\right) = e^{-\lambda t}$$

Remember, a is the initial time, so the probability that a patient will have to wait t minutes more given s minutes have already passed waiting is the same as waiting for t minutes initially. This is the **memoryless property** of the exponential distribution.

Relation with a Poisson process

Let Y be a Possion process with rate λ. Then in time t we expect λt events. What is the probability that we have to wait at least x until the 1^{st} Poisson event. Let X be the random variable which is the waiting time until 1^{st} Poisson event.

$$P(X > x) = P(Y = 0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x}$$

$$F(x) = P(X \leq x) = 1 - e^{-\lambda x}, \qquad x \geq 0$$

Example 2

A radioactive source is emitting α-particles as a Poisson process at a rate of 8 particles per minute. What is the probability that the interval between the emissions of two particles will be more than 40 sec?

$$P\left(X > \frac{2}{3}\right) = 1 - F\left(\frac{2}{3}\right) = 1 - \left(1 - e^{-8\cdot\frac{2}{3}}\right) = 0.00483$$

Example 3

Let X denote the time between detections of a particle with a Geiger counter and let x be exponential with $E(x) = 0.125\,\text{min}$

a. What is the probability that we will detect a particle within the first 10 sec ?

b. A minute has passed without detecting a particle. What is he probability that we will detect ore within the next 10 sec?

Solution

a. $P\left(x < \frac{1}{6}\right) = F\left(\frac{1}{6}\right) = 1 - e^{-\frac{1}{0.12}5\cdot\frac{1}{6}} = 1 - e^{-\frac{8}{6}} = 0.7364$

b.

$$\begin{aligned} P\left(x < 1\frac{1}{6} \,\middle|\, x > 1\right) = \frac{p\left(1 < x < 1\frac{1}{6}\right)}{P(x > 1)} &= \frac{F(1\frac{1}{6}) - F(1)}{1 - F(1)} \\ &= \frac{(1 - e^{-8.7/6}) - (1 - e^{-8})}{1 - (1 - e^{-8})} \\ &= \frac{e^{-8} - e^{-8.7/6}}{e^{-8}} \\ &= 1 - e^{8\cdot 1/6} \\ &= 0.7364 \end{aligned}$$

Remark
Remember: exponential random variables have lack of memory property.