§17. Normal Approximation to the Binomial and Poisson Distribution

Normal Approximation to the Binomial

Let $X \sim \text{Binom}(n,p)$ then

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately $N(0,1)$

Remark
The normal approximation to the Binomial is really an instance of the CLT in action.

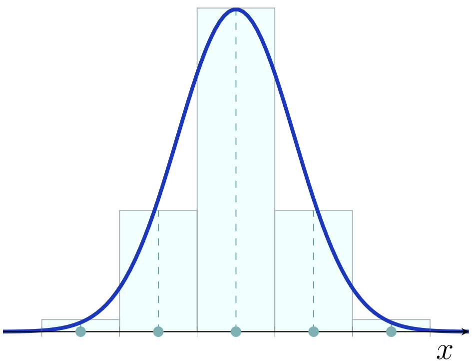
To approximate a binomial probability with a normal probability a **continuity correction** is applied as follows

$$P_{binom}(x \le X) = P_{normal}(x - 0.5 \le X) = P_{normal}\left(\frac{x - np - 0.5}{\sqrt{np(1-p)}} \le Z\right)$$

$$P_{binom}(X \le x) = P_{normal}(X \le x + 0.5) = P_{normal}\left(Z \le \frac{x - np + 0.5}{\sqrt{np(1-p)}}\right)$$

This approximation is acceptable for $np > 5, n(1-p) > 5$.

In the discrete distributions, the probability areas (bars) are centred on the value.



Example 1

In a communication channel the number of bits received in error can be modelled by a binomial random variable. Say $p = 10^{-5}$. If 16 million bits are transmitted, approximate the probability that 150 or fewer errors occur.

Solution

$$P_{binom}(X \leq 150) = \sum_{x=0}^{150} {}_{16000000}C_x \, (10^{-5})^x (1 - 10^{-5})^{16000000-x}$$

Normal approximation:

$\mu = np = 160$

$\sigma = \sqrt{np(1-p)} = \sqrt{160(1-10^{-5})}$

$$P(X \leq 150) \approx P\left(Z \leq \frac{150.5 - 160}{\sqrt{60(1-10^{-5})}}\right) = P(Z < -0.75) = 0.2266$$

Example 2

10% of humans are left handed. Compute and also approximate the probability that in a group of 50 people:

a. There will be three lefties or less.

b. There will be precisely three lefties.

Solution

a. $P(X \leq 3) = \sum_{x=0}^{3} {}_{50}C_x (0.1)^x (0.9)^{50-x} = 0.2503$

Approximation:

$\mu = (0.1)50 = 5$

$\sigma = \sqrt{5(0.9)} = 2.1213$

$$P(X \leq 3) \approx P\left(Z \leq \frac{3.5 - 5}{2.1213}\right) = P(Z \leq -0.71) = 0.2389$$

b. $P(X=3) = {}_{50}C_3\,(0.1)^3(0.9)^{47} = 0.1386$

$$P(X=3) \approx P\left(Z \leq \frac{3.5-5}{2\cdot 1213}\right) - P\left(Z \leq \frac{2.5\cdot 5}{2.1213}\right) = 0.2389 - 0.1150$$
$$= 0.1199$$

Normal Approximation to the Poisson

If $X \sim \text{Poissou}(\lambda)$ then $Z = \dfrac{X-\lambda}{\sqrt{\lambda}}$ is approximately normal. Continuity correction improves the approximation. The approximation is acceptable for $\lambda > 5$.

Example 3

The number of phages attached to bacteria living in sea water is on average 32 per bacteria. If a random sea water bacteria is selected what is the probability that there will be more than 36 phages attached to it?

Solution

$$P_{Poisson}(X > 36) \approx P_{normal}\left(Z > \frac{36.5-32}{\sqrt{32}}\right) = P_{normal}(Z > 0.80)$$
$$= 1 - 0.7881$$
$$= 0.2119$$

Exact Poisson: 0.2099