§19. Gamma Random Variables

The time to the first event in a Poisson process is exponentially distributed. What is the distribution of the time until the r^{th} event?

Example 1

In a datacenter the main number of failures per hour is 0.0001. Let X be the time until 4^{th} failure occurs. Determine $P(X < 40000)$.

Solution

Observe that $P(X > 40000) = P(N \le 3)$ where N has a Poisson distribution with $E(N) = 40000(0,0001) = 4$ per 40000 h

$$P(X > 40000) = P(N \le 3) = \sum_{k=0}^{3} \frac{e^{-4} \cdot 4^k}{k!} = 0.433$$
$$P(X < 40000) = 1 - 0.433 = 0.567$$

This example can be generalized to show that if x is the time until the r^{th} event in a Poisson process with rate λ, then

$$P(X > x) = \sum_{k=0}^{r-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!} \quad \text{and} \quad P(X < x) = 1 - \sum_{k=0}^{r-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$$

Definition 1

Consider the random variable X which denotes the time to r^{th} event in a Poisson process with rate λ.

Then X is called **Erlang random variable**, $X \sim \text{Earlang}(\lambda, r)$ and has cpf

$$F(x) = 1 - \sum_{k=0}^{r-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$$

Let's compute the pdf of the Erlang

$$\begin{aligned} p(X) &= F'(X) \\ &= -\sum_{k=0}^{r-1} \frac{(-\lambda)e^{-\lambda x}(\lambda x)^k}{k!} - \sum_{k=0}^{r-1} \frac{e^{-\lambda x} \cdot (\lambda k) \cdot (\lambda x)^{k-1}}{k!} \\ &= \lambda e^{-\lambda x} \left[\sum_{k=0}^{r-1} \frac{(\lambda x)^k}{k!} - \sum_{m=0}^{r-2} \frac{(\lambda x)^m}{m!} \right] \qquad \text{the } (r-1) \text{ terms cancel} \end{aligned}$$

$$p(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!} \quad ; \quad x > 0, \quad r = 1, 2, 3, \ldots$$

Now we know $(r-1)! = \Gamma(r)$, so naturally we can extend this density to non-integer values of r.

Definition 2

A **Gamma random variable** is a random variable X with pdf

$$p(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)} \qquad x > 0$$

$X \sim \text{Gamma}(\lambda, r)$. The parameters λ and r a called **scale** and **shape** parameters. The expected valve and the variance are

$$E(x) = \frac{r}{\lambda} \qquad \text{Var}(X) = \frac{r}{\lambda^2}$$

Example 2

Suppose $X \sim \text{Gamma}(2.3, 4.7)$. Use excel to compute $P(X < 2)$.

Solution

$$\alpha = r \qquad \beta = \frac{1}{\lambda} \qquad P(X < 2) = 0.5431$$

When $r = \dfrac{k}{2}$ and $\lambda = \dfrac{1}{2}$, the Gamma random variable is called χ^2 **random variable** with k degrees of freedom. $X \sim \chi^2(k)$ The pdf is

$$\boxed{p(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}}$$

One reason χ^2 is very important: Let $Z_1, \dots, Z_k$ be independent, standard normal random variables. Then

$$Q = \sum_{i=1}^{u} Z_i^2 \quad \text{is} \quad Q \sim \chi^2(k)$$

From the results on the more general Gamma family we have:

$$E(Q) = \frac{k/2}{1/2} = k \qquad \mathrm{Var}(Q) = \frac{k/2}{(1/2)^2} = 2k$$