§2. Theorems of Probability

The Principle of Indifference

The principle of indifference is a rule for assigning (Bayesian epistemic) probabilities. It states that:

In the absence of any relevant evidence, agents should assign equal probabilities to all possible outcomes under consideration.

Example 1

Toss a coin: $P(T) = \frac{1}{2} = P(H)$

Toss a die: $P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$

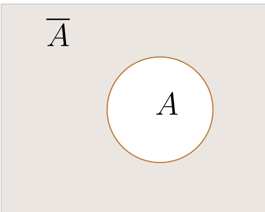
Select a card at random from a shuffled deck: $P(\mathsf{K}\heartsuit) = \frac{1}{52}$

Remark The principle of indifference has no meaning under the frequencies interpretation of probability.

Theorem 1

$$P(\overline{A}) = 1 - P(A)$$

Proof.



$$A \cup \overline{A} = S$$
$$P(A) + P(\overline{A}) = P(S) = 1$$

□

Example 2

Toss two dice. What is the probability that they show different numbers?

Solution

A = they show the same numbers

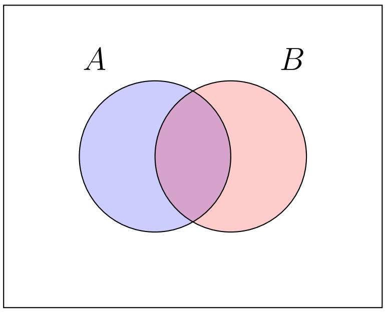
$$A = \{\,(1,1),(2,2),\ldots,(6,6)\,\}$$

$$\therefore \quad P(\,\overline{A}\,) = 1 - P(A) = 1 - \frac{6}{36} = \frac{5}{6}$$

Theorem 2: Addition Law for Two Events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof.



$$A = (\,A \backslash B\,) \cup (\,A \cap B\,)$$
$$B = (\,B \backslash A\,) \cup (\,A \cap B\,)$$

$$A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$$
$$P(A) = P(A \backslash B) + P(A \cap B)$$
$$P(B) = P(B \backslash A) + P(A \cap B)$$

$$\begin{aligned} \therefore \qquad P(A \cup B) &= P(A \backslash B) + P(B \backslash A) + P(A \cap B) \\ &= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] + P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

□

Example 3

A card is pulled at random from a well shuffled deck. What is the probability that the card is a face card or a spade?

Solution

F = a face card
S = a spade

$$P(F \cup S) = P(F) + P(S) - P(F \cap S) = \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{11}{26}$$

Example 4

A card is pulled. What is the probability that it is not a face card or a spade?

Solution

F = a face card
S = a spade

$$P(\overline{F}\cap\overline{S}) = P(\overline{F\cup S}) = 1 - P(F\cup S) = 1 - \frac{11}{26} = \frac{15}{26}$$

Remark
Recall: DeMorgan Laws

i. $\overline{A\cup B} = \overline{A}\cap\overline{B}$

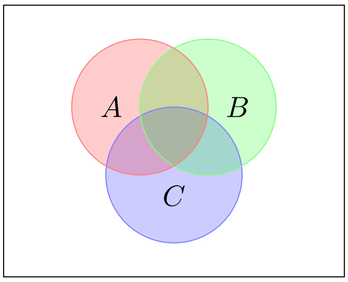
ii. $\overline{A\cap B} = \overline{A}\cup\overline{B}$

Theorem 3: Addition Law for Multiple Events

Three events:

$$\begin{aligned} P(A\cup B\cup C) = P(A) + P(B) + P(C) \\ - P(A\cap B) - P(B\cap C) - P(A\cap C) \\ + P(A\cap B\cap C) \end{aligned}$$

n–events: add the probabilities for the intersections of odd number of events and subtract the even ones.



Example 5

Toss two dice. What is the probability that the sum is 7, or the difference between the two dice is ±1, or exactly one die shows a 4?

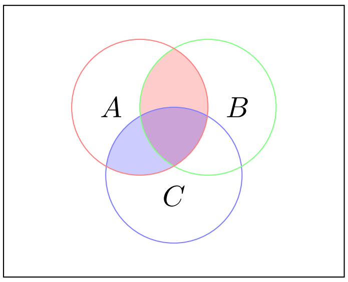
Solution 1

A = sum is 7
B = the difference is ±1
C = one die shows a 4

$$\begin{aligned} P(A\cup B\cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A\cap B) - P(B\cap C) - P(A\cap C) \\ &\quad + P(A\cap B\cap C) \\ &= \frac{6}{36} + \frac{10}{36} + \frac{11}{36} - \frac{2}{36} - \frac{2}{36} - \frac{4}{36} + \frac{2}{36} \\ &= \frac{21}{36} \end{aligned}$$

Distributive Identities

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$