§20. Functions of Random Variables

Example 1

Let $X \sim \text{Uniform}(0,1)$. A square with length X is formed. What is the expected area?

$E(X) = \frac{1}{2}$

$$E(Area) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \qquad \text{Wrong!!!} \qquad E(X^2) \neq (E(X))^2$$

$$E(Area) = E(X^2) = \int_0^1 x^2 \cdot p(x)\, dx = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

Example: Cont'd

What is the probability that the area will fall into interval $\left(\frac{1}{2}, \frac{3}{4}\right)$?

$$P\left(\frac{1}{2} \leq x^2 \leq \frac{3}{4}\right) = P\left(\frac{1}{\sqrt{2}} \leq x \leq \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} = 0.15892$$

Example 2

Let $X \sim \exp(\lambda)$. Compute $E(\sqrt{X})$.

$$\begin{aligned} E(\sqrt{X}) &= \int_0^\infty \sqrt{x}\lambda e^{-\lambda x} dx \qquad ; \qquad u = \lambda x \\ &= \frac{1}{\sqrt{\lambda}} \int_0^\infty \sqrt{u} e^{-u} du \\ &= \frac{1}{\sqrt{\lambda}} \cdot \Gamma\left(\frac{3}{2}\right) \\ &= \frac{1}{\sqrt{\lambda}} \cdot \frac{\sqrt{\pi}}{2} \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \end{aligned}$$

We need to learn how to deal with functions of random variables.

$$Y = f(X), \qquad p(x) - \text{ known, what is } p(y) = ?$$

Let's start with some examples.

Example 3

Suppose that $X \sim \text{Uniform}(3,5)$ so that $p(x) = \frac{1}{2}$ and let $Y = \frac{X-2}{3}$ Find the pdf of Y.

$$F_y(y) = P(Y \leq y) = P\left(\frac{x-2}{3} \leq y\right) = P(x \leqslant 3y+2) = F_x(3y+2)$$

$$\begin{aligned} p(y) &= F_y'(y) \\ &= \frac{dF_x(3y+2)}{dy} = F_x'(3y+2)\cdot 3 = p_x(3y+2)\cdot 3 = \frac{1}{2}\cdot 3 = \frac{3}{2} \end{aligned}$$

Thus, $p(y) = \frac{3}{2}$ on $\left(\frac{1}{3}, 1\right)$

Now let's try this in general: let $Y = H(X)$ where H is monotonous (and hence invertible) function.

$$F_Y(y) = P(Y \leq y) = P(H(X) \leqslant y) = P\left(X \leqslant H^{-1}(y)\right) = F_x\left(H^{-1}(y)\right)$$

$$\begin{aligned} p(y) &= \frac{dF_y(y)}{dy} \\ &= \frac{dF_x\left(H^{-1}(y)\right)}{dy} \\ &= F_x'\left(H^{-1}(y)\right)\cdot\frac{dH^{-1}(y)}{dy} = p(x)\cdot\frac{dx}{dy} \qquad \text{with } x = H^{-1}(y) \end{aligned}$$

$$\boxed{p(y)dy = p(x)dx}$$

Example 4

Let $X \sim N(0,1)$ $\quad Y = e^X$ is log normal. Determine its pdf.

$$p(y) = p(x)\cdot\frac{dx}{dy} = p(\ln y)\cdot\frac{d\ln y}{dy} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(\ln y)^2}{2}}\cdot\frac{1}{y}$$

Non-example:

$X \sim N(0,1)$, $\qquad Y = X^2 \quad \leftarrow \quad$ This is not a monotone function.

$$F_y(y) = p(y \leqslant y) = p\left(x^2 \leq y\right) = P(-\sqrt{y} \leqslant x \leqslant \sqrt{y}) = 2(p(x \leq \sqrt{y}) - 0.5) = 2F_x(\sqrt{y}) - 1$$

$$p(y) = \frac{dF_y(y)}{dy} = 2\frac{dF_x(\sqrt{y})}{dy} = 2F_x'(\sqrt{y})\frac{d\sqrt{y}}{dy} = 2p(x)\frac{dx}{dy} \qquad ; \qquad x = \sqrt{y}$$

$$p(y) = 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \cdot \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \qquad ; \qquad y > 0 \text{ which is } \chi^2(1).$$

Let's go back to $Y = H(X)$ with H monotone. Let's compute the expected value of Y via $p(x)$:

$$E(y) = \int_{-\infty}^{\infty} yp(y)\,dy = \int_{-\infty}^{\infty} H(x) \cdot p(x) \cdot \frac{dx}{dy}\,dy = \int_{-\infty}^{\infty} H(x)p(x)\,dx$$

How about, when H is not monotone?

Consider again the case where $Y = X^2$

$$\begin{aligned} E(Y) = \int_{-\infty}^{\infty} yp(y)\,dy &= \int_{0}^{\infty} yp(y)\,dy \\ &= \int_{0}^{\infty} x^2 \cdot 2 \cdot p(x)\frac{dx}{dy}\,dy \qquad ; \quad x = \sqrt{y} \\ &= 2\int_{0}^{\infty} x^2 p(x)\,dx \\ &= \int_{-\infty}^{\infty} x^2 p(x)\,dx \end{aligned}$$

$$E(Y) = \int_{-\infty}^{\infty} H(x)p(x)dx$$

Theorem: Law of the unconsciuos statistician (LOTUS)

If X is a random variable and $H(x)$ is a (deterministic) function then

$$E(H(X)) = \int_{-\infty}^{\infty} H(x)p(x)dx$$

Example 5

Let X have pdf

$$p(x) = \begin{cases} \dfrac{5}{33}x^4 & ; \quad -1 \le x < 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

This is a valid density:

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_{-1}^{2} \frac{5}{33} \cdot x^4\,dx = \frac{5}{33} \cdot \frac{x^5}{5}\bigg|_{-1}^{2} = 1$$

Let $Y = X^2$. Determine $E(Y)$ and $p(y)$.

Solution

$F(y) = P(Y \leqslant y) = P\left(X^2 \leqslant y\right)$ and now we have two cases $y \in (0,1)$ and $y \in (1,4)$.

Case 1: $y \in (0,1)$

$$\begin{aligned} F(y) &= P\left(x^2 \le y\right) \\ &= P(-\sqrt{y} \leqslant x \leqslant \sqrt{y}) \\ &= \frac{5}{33}\int_{-\sqrt{y}}^{\sqrt{y}} x^4 dx = \frac{5}{33} \cdot \frac{x^5}{5}\bigg|_{-\sqrt{y}}^{\sqrt{y}} = \frac{2}{33}y^{5/2} \qquad ; \qquad F(1) = \frac{2}{33} \end{aligned}$$

Case 2: $y \in (1,4)$

$$\begin{aligned} F(y) &= \frac{2}{33} + P\left(1 \le x^2 \le y\right) \\ &= \frac{2}{33} + P(1 \le x \le \sqrt{y}) \\ &= \frac{2}{33} + \frac{5}{33}\int_{1}^{\sqrt{y}} x^4 dx = \frac{2}{33} + \frac{5}{33} \cdot \frac{x^5}{5}\bigg|_{1}^{\sqrt{y}} = \frac{1}{33} + \frac{1}{33}y^{5/2} \end{aligned}$$

$$F(y) = \begin{cases} 0 & ; \quad y \le 0 \\ \frac{2}{33}y^{5/2} & ; \quad 0 < y \le 1 \\ \frac{1}{33}(1 + y^{5/2}) & ; \quad 1 < y \le 4 \\ 1 & ; \quad y > 4 \end{cases} \qquad P(y) = \begin{cases} 0 & ; \quad y \le 0 \\ \frac{2}{33}y^{3/2} & ; \quad 0 < y \le 1 \\ \frac{5}{66}y^{3/2} & ; \quad 1 < y \le 4 \\ 0 & ; \quad y > 4 \end{cases}$$

$$
\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} y p(y)\, dy \\
&= \int_0^1 y \cdot \frac{5}{33} y^{3/2} dy + \int_1^4 y \cdot \frac{5}{66} y^{3/2}\, dy = \frac{5}{33} \cdot \frac{y^{7/2}}{7/2}\bigg|_0^1 + \frac{5}{60} \cdot \frac{y^{7/2}}{7/2}\bigg|_1^4 \\
&= \frac{645}{231} \\
&= \frac{215}{77}
\end{aligned}
$$

$$
E(Y) = \int_{-\infty}^{\infty} x^2 p(x) dx = \int_{-1}^{2} x^2 \cdot \frac{5}{33} \cdot x^4 dx = \frac{5}{33} \cdot \frac{x^7}{7}\bigg|_{-1}^{2} = \frac{645}{231} = \frac{215}{77}
$$

Inverse transform sampling

Here is an explanation of why inverse transform sampling works: Let X be a random variable with monotone cpf $F(x)$. Let $Y = F(X)$.

$$
\frac{dy}{dx} = F'(x) = p(x) \quad ; \quad \frac{dx}{dy} = \frac{1}{p(x)}
$$

$$
p(y) = p(x) \cdot \frac{dx}{dy} = p(x) \cdot \frac{1}{p(x)} = 1 \qquad \Rightarrow \qquad Y \sim \text{Uniform}(0,1)
$$