§21. Moment Generating Functions

Definition 1

Let X be a random variable. The n^{th} **moment of** X is $E(X^k)$

$$E(X^k) = \sum_x x^k\, p(x) \qquad \text{if } X \text{ is discrete.}$$

$$E(X^k) = \int_{-\infty}^{\infty} x^k\, p(x)\, dx \qquad \text{if } X \text{ is continuous}$$

Example 1

$E(X)$ this is the expected value.
$E(X^2)$ this is related to the variance.

Definition 2

The **moment generating function** of a random variable X is

$$M_X(t) = E(e^{tx})$$

provided this expectation exists. Thus

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} p(x) dx \qquad \text{(Uiuqe for } X\text{)}$$

$M_X(t)$ does indeed "generate" the moments of X:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} p(x)\, dx \\ &= \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right) p(x)\, dx \\ &= \int_{-\infty}^{\infty} 1 \cdot p(x)\, dx + \int_{-\infty}^{\infty} tx \cdot p(x)\, dx + \int_{-\infty}^{\infty} \frac{t^2x^2}{2!} \cdot p(x)\, dx + \cdots \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) \end{aligned}$$

In the moment generating function the coefficient of the n^{th} moment $E(X^k)$ is $\frac{t^k}{k!}$. Notice that

$$M_X(0) = 1 \qquad M'_X(0) = E(X) \qquad M''_X(0) = E(X^2), \qquad \dots$$

Example 2

Let $X \sim \text{Uniform}(0, 1)$

$$M_X(t) = \int_0^1 e^{tx} \cdot 1 \, dx = \frac{1}{t}(e^t - 1) = \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!}$$

$$\Rightarrow E(X^k) = \frac{1}{k+1}$$

Indeed:

$$E(X^k) = \int_0^1 x^k \cdot 1 \, dx = \frac{1}{k+1}$$

Example 3

Let $X \sim \exp(\lambda)$ i.e. $p(x) = \lambda e^{-\lambda x}$, $\lambda > 0$

$$M_X(t) = \int_0^\infty e^{tx} \cdot e^{-\lambda x} \, dx = \lambda \int_0^\infty e^{(t-\lambda)x} \, dx = \frac{\lambda}{t - \lambda} \, e^{(t-\lambda)x}\Big|_0^\infty$$

Assume $t < \lambda$

$$M_X(t) = \frac{\lambda}{t - \lambda}(0 - 1) = \frac{\lambda}{\lambda - t}$$

Example 4

Let $X \sim N(0, 1)$

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{x^2}{2}} \, dx$$

$$tx - \frac{x^2}{2} = \frac{t^2}{2} - \frac{1}{2}(x - t)^2$$

$$M_X(t) = e^{\frac{t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} \, dx = e^{\frac{t^2}{2}} \qquad \boxed{M_X(t) = e^{\frac{t^2}{2}}}$$

Notice that

$$M_X(t) = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \cdots +$$

$$\Rightarrow E(X^k) = 0 \text{ for } k \text{ odd} \quad ; \quad E(X^{2k}) = \frac{(2k)!}{k!2^k} \text{ for } k = 1, 2, 3, \ldots$$

Theorem 1

a. $M_{cX}(t) = M_X(ct)$

b. $M_{X+c}(t) = e^{ct} M_X(t) \quad ; \quad c \neq 0$

Proof.

a. $M_{cX}(t) = \int_{-\infty}^{\infty} e^{t(cx)}\, p(x)\, dx = \int_{-\infty}^{\infty} e^{(ct)x}\, p(x)\, dx = M_x(ct)$

b. $M_{c+X}(t) = \int_{-\infty}^{\infty} e^{t(x+c)}\, p(x)\, dx = e^{tc} M_x(t)$

□

Corollary

$$X \sim N(\mu, \sigma) \quad ; \quad X = \mu + Z\sigma, \quad Z \sim N(0,1) \Rightarrow \boxed{M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

Example 5

$M_t(x) = e^{\frac{3t}{4} + \frac{t^2}{3}}$ Identify the random variable.

Solution

$x \sim N\left(\frac{3}{4}, \sqrt{\frac{2}{3}}\right)$

Theorem 2

If X, Y are independent random variables, then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Proof.

For independent random variables X and Y, $P(X,Y) = P(x)P(y)$

$$
\begin{aligned}
M_{X+Y}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(x+y)t} p(x,y)\,dx\,dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{xt}e^{yt}\,p(x)\,p(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty} e^{xt}\,p(x)\,dx \cdot \int_{-\infty}^{\infty} e^{yt}\,p(y)\,dy \\
&= M_X(t)\cdot M_Y(t)
\end{aligned}
$$

□

Example 6

Let $X \sim N(-1,3)$ and $Y \sim N(2,4)$ and X, Y independent.

$$
\begin{aligned}
M_X(t) &= e^{-t+\frac{9t^2}{2}} \\
M_Y(t) &= e^{2t+\frac{16t^2}{2}}
\end{aligned}
$$

$$
M_{X+Y}(t) = M_X(t)\cdot M_Y(t) = e^{t+\frac{25t^2}{2}} \qquad \Rightarrow \qquad X+Y \sim N(1,5)
$$

In general, let $X \sim N(\mu_x,\sigma_x)$, and $Y \sim N(\mu_y,\sigma_y)$ and X, Y−independent.

$$
M_{X+Y}(t) = M_X(t)\cdot M_Y(t) = e^{\mu_x t+\frac{\sigma_x^2 t^2}{2}} \cdot e^{\mu_y t+\frac{\sigma_y^2 t^2}{2}} = e^{\mu_x+\mu_y+\frac{t^2(\sigma_x^2+\sigma_y^2)}{2}}
$$

$$
\begin{aligned}
E(X+Y) &= E(X)+E(Y) \\
\mathrm{Var}(X+Y) &= \mathrm{Var}(X)+\mathrm{Var}(Y)
\end{aligned}
$$