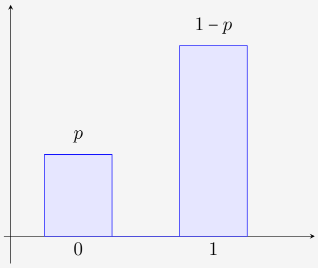
§22. Sums of Random Variables

In many contexts the random variables of interest is a sum of "more elementary" random variables.

Example 1

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables

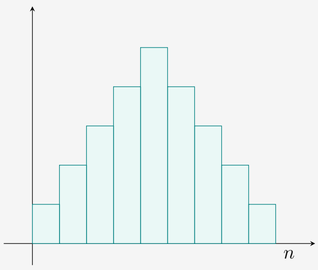
$$X_i = \begin{cases} 1 & ; \quad \text{with probability, } \; p \\ 0 & ; \quad \text{with probability, } \; 1-p \end{cases}$$



Let $X = X_1 + X_2 + X_3 + \cdots + X_n$ then

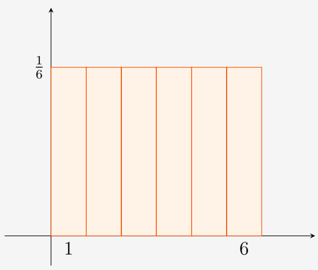
$$P(X = x) =_n C_x p^n (1-p)^{n-x}$$

i.e. $X \sim \text{Binom}(n, p)$

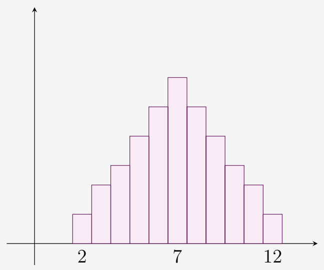


Example 2

X, Y – Discrete Uniform$[1, 6]$



$X+Y$	2	3	4	5	6	7	8	9	10	11	12
$P(X+Y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{63}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



Example 3

Let $X_i \sim \exp(1)$. Then from before

$$M_{x_i}(t) = \frac{1}{1-t}$$

Let $X = X_1 + X_2 + X_3 + \cdots + X_n$ be the sum of n independent exponentials.

$$M_X(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) = \frac{1}{(1-t)^n}$$

Notice that $E(X) = n \cdot 1 = n$ and $\mathrm{Var}(X) = n \cdot 1 = n \Rightarrow \sigma(X) = \sqrt{n}$. Consider the random variable

$$Z = \frac{X - E(X)}{\sigma(X)} = \frac{X-n}{\sqrt{n}}$$

For its moment generating function we have:

$$M_{Z(t)} = M_{\frac{X-n}{\sqrt{n}}} = e^{-t\sqrt{n}} M_X\left(\frac{t}{\sqrt{n}}\right) = e^{-t\sqrt{n}} \cdot \frac{1}{(1-\frac{t}{\sqrt{n}})^n}$$

↑

linear transformation

$$= e^{-t\sqrt{n}}\left(1 - \frac{t}{\sqrt{n}}\right)^{-n}$$

Let's investigate the limit $n \to \infty$. To simplify the computation we will evaluate the limit of $\ln(M_z(t))$ as $u \to \infty$.

$$\lim_{n\to\infty} \ln(M_z(t)) = \lim_{n\to\infty} \ln\left\{e^{-t\sqrt{n}}\left(1 - \frac{t}{\sqrt{u}}\right)^{-4}\right\}$$

$$= \lim_{n\to\infty}\left\{-t\sqrt{u} - n\ln\left(1 - \frac{t}{\sqrt{u}}\right)\right\} = *$$

The Taylor expansion of $\ln(1+x)$ is:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$
\begin{aligned}
* &= \lim_{n\to\infty}\left\{-t\sqrt{n} - n\left[-\frac{t}{\sqrt{n}} - \frac{t^2}{n\cdot 2} - \frac{t^3}{n^{3/2}\cdot 3} - \frac{t^4}{n^2\cdot 4} - \cdots\right]\right\} \\
&= \lim_{n\to\infty}\left\{\cancel{-t\sqrt{n}} + \cancel{t\sqrt{n}} + \frac{t^2}{2} + \frac{t^3}{\sqrt{n}\cdot 3} + \frac{t^4}{n\cdot 4} + \cdots\right\} \\
&= \frac{t^2}{2}
\end{aligned}
$$

Thus,

$\lim_{n\to\infty} M_z(t) = e^{t^2/2} \leftarrow$ the moment generating function of standard normal and Z is standard normal in the limit (asymptotically).