§25. Sampling Distribution of the Mean; Confidence Intervals and Hypothesis Testing

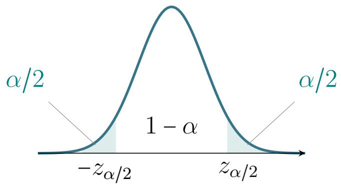
Remember:

- If $X \sim N(\mu, \sigma)$ then $\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ for every n.
- If X is any random variable to which the CLT applies with $E(X) = \mu$ and $\text{Var}(x) = \sigma^2$ Then $\overline{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ in the limit $n \to \infty$. FAPP $n > 30$ is sufficient.

Case 1

σ is known and either $X \sim N(\mu, \sigma)$ or $n > 30$

$$p\left(-z_{\alpha/2} \leq \frac{\overline{x} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) = 1 - \alpha$$



A $100(1-\alpha)$ confidence interval for $\overline{x}$ is

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

> **Example 1**
>
> A sample of 10 observations from $N(\mu, \sigma = 6)$ gave a sample mean $\bar{x} = 28.45$. Compute the 90%, 95%, and 99% confidence interval for μ.

Solution

90% confidence interval

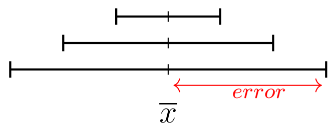
$$28.45 - 1.645\frac{6}{\sqrt{10}} \le \mu \le 28.45 + 1.645\frac{6}{\sqrt{10}}$$
$$25.3288 \le \mu \le 31.5712$$

95% confidence interval

$$28.45 - 1.96\frac{6}{\sqrt{10}} \le \mu \le 28.45 + 1.96\frac{6}{\sqrt{10}}$$
$$24.7311 \le \mu \le 32.1688$$

99% confidence interval

$$28.45 - 2.576\frac{6}{\sqrt{10}} \le \mu \le 28.45 + 2.576\frac{6}{\sqrt{10}}$$
$$23.5624 \le \mu \le 33.34$$



High precision → Low confidence
Low precision → High confidence

Example 2

How large of a sample must be selected from a normal random variable with $\sigma = 12$ in order to estimate μ with error at most 2 units and with 95% confidence?

Solution

$$E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \Rightarrow \quad n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2$$
$$= \left(\frac{(1.96)\cdot(12)}{2}\right)^2 = 138.3 \quad \Rightarrow \quad 139$$

$100(1-\alpha)\%$ confidence bounds are

Lower Confidence Bound

$$\mu \geqslant \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}$$

Upper Confidence Bound

$$\mu \leqslant \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Example 3

A sample of 45 observations from $N(\mu, \sigma = 9)$ gave a sample mean $\bar{x} = 17.2$. Compute a 99% confidence lower bound.

Solution

$$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{u}} = 17.2 - 2.326 \frac{9}{\sqrt{45}} = 14.08 \leqslant \mu \quad ; \quad \text{with 99\% confidence}$$

Hypothesis Testing

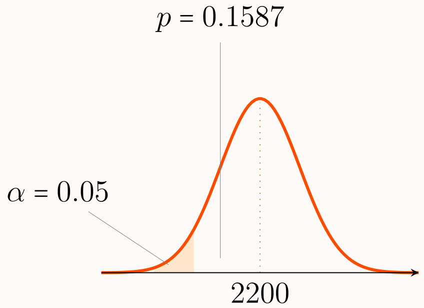
Since $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ is (approximately) $N(0,1)$ we can use it to test $H_0 : \mu = \mu_0$ versus $H_1 : \mu < \mu_0$, $H_1 : \mu > \mu_0$, or $H_1 : \mu \neq \mu_0$.

Example 4

In the past the tensile strength of steel rods from a manufacturer had a mean $\mu_0 = 2200$ N with $\sigma = 460$ N. Suppose the tensile strength is normal.

Recently we sampled 25 steel rods and found mean tensile strength $\bar{x} = 2108$ N. Test $H_0 : \mu = 2200$ versus $H_1 : \mu < 2200$ at the $\alpha = 0.05$ level of significance. Report the $p - value$.

Solution



$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{2108 - 2200}{460/\sqrt{25}} = -1$$

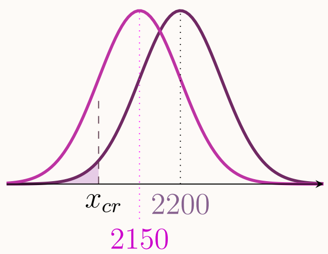
$$\begin{aligned} p - \text{value} &= p(z < -1) \\ &= 0.1587 > 0.05 = \alpha \end{aligned}$$

Fail to reject H_0. This sample does not provide enough evidence to claim that the tensile strength has decreased.

Example: Cont'd

If the true mean tensile strength is $\mu_t = 2150$ N what is the probability for Type II error for the test above. What is the power of the test?

Solution



$$x_{cr} = 2200 - 1.645 \cdot \frac{(460)}{\sqrt{25}} = 2048.66$$

$$z = \frac{2048.66 - 2150}{466/\sqrt{25}} = -1.10$$

$\beta = p(z > -1.10) = 0.8643$; Power $= 0.1357$

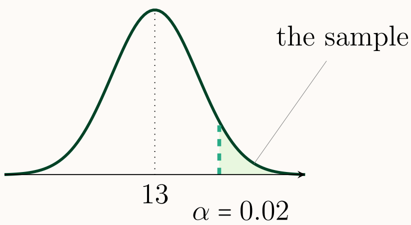
Example 5

A manufacturer of sprinkler systems used for fire protection of office buildings claims the activation temperature is 130°C. A sample of 9 such systems yields a sample average activation temperature of 131.08°C. If the distribution of activation times is Gaussian with standard deviation of 1.5°C, does this data contradict the manufacturers claim at significance level $\alpha = 0.02$? Test $H_0 : \mu = 130$ versus $H_1 : \mu > 13$.

Solution

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{131.08 - 130}{1.5/\sqrt{9}} = 2.16$$

$$p - value = p(z > 2.16)$$
$$= 0.0154 < \alpha = 0.02$$



Reject H_0. Accept H_1. The activation temperature is more than 130°C

Case 2

σ is not known, but $n > 40$. Use s as an estimator (in place) of σ. This approximation works for both confidence intervals and hypothesis testing.

Example 6

A sample of 48 ATM client transaction times has sample mean of 261 s will sample standard deviation of 22 s. Test $H_0 : \mu = 270$

versus $H_1 : \mu < 270$ at $\alpha = 0.05$ level of significance.

Solution

$H_1 : \mu < 270$ at $\alpha = 0.05$ level of significance.

$$z = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{261 - 270}{22/\sqrt{48}} = -2.83$$

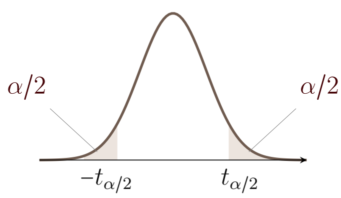
$p - value = p(z < -2.83) = 0.0023 < 0.05 = \alpha$

Reject H_0. Accept H_1. The client transaction times are less than 270 s.

Case 3

$X \sim N(\mu, \sigma)$; σ-not known; small sample $n \leq 40$.

Let $X_1, \ldots, X_n$ be a random sample from $X \sim N(\mu, \sigma)$. Let $\overline{x}$ be the sample mean and s the sample standard deviation the random variable $T = \dfrac{\bar{X} - \mu}{s/\sqrt{n}}$ is student $(n-1)$



$$p\left(-t_{\alpha/2} \leq \frac{\bar{x} - \mu}{s/\sqrt{n}} \leq t_{\alpha/2}\right) = 1 - \alpha$$

A $100(1-\alpha)\%$ CI for μ based on a random sample of size n from a normal distribution is

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{x}} \leq \mu \leq \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

Example 7

Ten randomly selected hot-dogs have the following fat content (in g)

25.2 21.3 22.8 17.0 29.8 21.0 25.5 16.6 20.9 19.5

Source: Sensory and Mechanical Assessment of The Quality of Frankfurters, J. of Texture Stud., 1990: 395-409.

Assuming that the fat content is Gaussian, construct a 95% CI for its mean.

Solution

$n - 1 = 10 - 1 = 9 \qquad t_{0.025} = 2.262$

$\bar{x} = \frac{1}{n}\sum x_i = 21.9 \qquad s^2 = \frac{1}{n-1}\sum (x_i - \bar{x})^2 = 17.09 \quad ; \quad s = 4.134$

$$21.9 - 2.262 \cdot \frac{4.134}{\sqrt{10}} \le \mu \le 21.9 + 2.262 \cdot \frac{4.134}{\sqrt{10}}$$

$$18.94 \le \mu \le 24.86 \text{ g} \quad ; \quad \text{with 95\% confidence.}$$

To test $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$, $H_1 : \mu < \mu_0$, or $H_1 : \mu \neq \mu_0$ small sample; use the statistic

$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}}$$

Example 8

Glycerol is a by-product of ethanol fermentation in wine and contributes to the sweetness, body and fullness of wines. The desired concentration is 4mg/mL A distributor of white wines wants to sell you 25 000 bottles. You test 5 bottles and find The following glycerol concentrations:

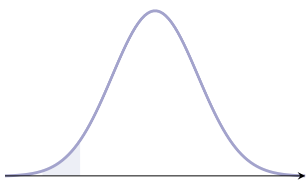
2.67 4.62 4.14 3.81 3.83

Assuming the glycerol concentration is normal, test $H_0 : \mu = 4$ versus $H_1 : \mu < 4$.

Solution

$\bar{x} = 3.814, \quad s = 0.718; \quad t = \dfrac{3.814 - 4}{0.718/\sqrt{5}} = -0.58$ with 4 d.f.

$p - value = p(t < -0.58) = 0.30$



Cannot reject H_0 at $\alpha = 0.05$ level of significance. This sample does not provide sufficient evidence to reject the offer.

Example 9

A city health department wishes to confirm that the mean bacteria count per unit volume at a lake is below 200 (safe to drink). Ten samples have the following bacteria counts:

175 190 205 193 184 207 204 193 196 180

Is there a cause for concern at $\alpha = 0.05$ level of significance?

Solution

Test $H_0 : \mu = 200$ versus $H_1 : \mu < 200$

$$\bar{x} = 192.7 \quad s = 10.81 \quad t = \frac{192.7 - 200}{10.81/\sqrt{10}} = -2.14 \quad 9 \text{ d.f.}$$

$$p - value = p(t < -2.14) = 0.0305 < 0.05 = \alpha$$

Reject H_0. Accept H_1. The bacteria count is within the safety level