§28. Joint Distributions

Most contexts have move than one random variable present. If we study them separately we will not capture how the variables interact and this is frequently the core research question.

Example 1

X - the number of bars on a cellphone
Y - the number of repetitions needed for an AB system to recognize a word

$p(x,y)$	$x=1$	$x=2$	$x=3$	$p(y)$
$y=4$	0.15	0.10	0.05	0.30
$y=3$	0.02	0.10	0.05	0.17
$y=2$	0.02	0.03	0.20	0.25
$y=1$	0.01	0.02	0.25	0.28
$p(x)$	0.20	0.25	0.55	1

This is an example of a bivariate probability distribution;

$$P(x,y) = P\left(X = x, Y = y\right)$$

Definition 1

The **joint pmf** of two discrete random variables X and Y is the function $\rho_{x,y}(x,y) = p(x,y)$ that satisfies

1. $p(x,y) \geq 0$
2. $\sum\limits_{x,y} p(x,y) = 1$
3. $p(x,y) = P(X = x, Y = y)$

Definition 2

The **joint pdf** for two continuous random variables satisfies

1. $p(x,y) \geq 0$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\, dx\, dy = 1$

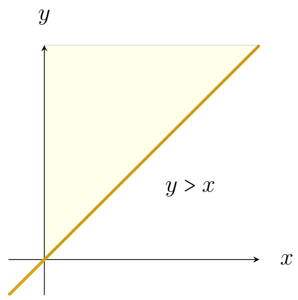
3. For any region R of $\mathbb{R}^2$,

$$P((X,Y)\in R) = \iint_R p(x,y)\,dx\,dy$$

Example 2

X - time your laptop connects to a server (in ms)
Y - time the server recognizes you as a valid user (in ms)

$$p(x,y) = 6\times 10^{-6}\exp(-0.001x-0.002y) \qquad 0<x<y$$



Let's check that this is a valid poff:

$$\begin{aligned}\iint_R p(x,y)\,dx\,dy &= \int_0^\infty dx\int_x^\infty dy\,6\times 10^{-6}\exp(-0.001x-0.002y)\\ &= 6\times 10^{-6}\int_0^\infty dx\left[\frac{e^{-0.001x-0.002y}}{-0.002}\right]_{y=x}^{y=\infty}\\ &= 3\times 10^{-3}\int_0^\infty dx e^{-0.001x-0.002x}\\ &= 1\end{aligned}$$

Let's compute the following 'joint' probability:

$$\begin{aligned}P(X\le 1000, Y\le 2000) &= \int_0^{1000} dx\int_x^{2000} dy\,6\times 10^{-6}\,e^{-0.002y-0.001x}\\ &= -3\times 10^{-3}\int_0^{1000} dx\left[e^{-0.002y-0.001x}\right]_{y=x}^{y=2000}\\ &= 3\times 10^{-3}\int_0^{1000} dx\left[e^{-0.003x}-e^{-4}e^{-0.001x}\right]\\ &= 3\times 10^{-3}\left[\frac{e^{-0.003x}}{(-0.003)}-e^{-4}\frac{e^{-6.001x}}{(-0.001)}\right]_{x=0}^{1000}\\ &= 3\times 10^{-3}\left[\frac{e^{-3}-1}{(-0.003)}-e^{-4}\frac{(e^{-1}-1)}{(-0.001)}\right]\\ &= 0.915\end{aligned}$$

How does this work in the discrete case?

Example 3: Bars on Cellphone

$$P(x\le 2, y\ge 3) = 0.15+0.1+0.02+0.1 = 0.37$$

Definition 3

If The joint pmf (pdf) of two random variables X and Y is $p(x, y)$ the **marginal pmf's (pdf's)** of X and Y are:

$$p_X(x) = \sum_y p(x, y) \quad ; \quad p_Y(y) = \sum_x p(x, y)$$

or

$$p_X(x) = \int_{-\infty}^{\infty} p(x, y)\, dy \quad ; \quad p_Y(y) = \int_{-\infty}^{\infty} p(x, y)\, dx$$

Example 4

The marginals of the cellphone bars example are in the margins.

Example 5

Laptop connecting to a server.

$$\begin{aligned} p_X(x) = \int_{-\infty}^{\infty} p(x, y) dy &= 6 \times 10^{-6} \int_x^{\infty} e^{-0.001x - 0.002y}\, dy \\ &= 0.003 e^{-0.003x} \end{aligned}$$

$$\begin{aligned} p_Y(y) = \int_{-\infty}^{\infty} p(x, y) dx &= 6 \times 10^{-6} \int_0^{y} e^{-0.001x - 0.002y}\, dx \\ &= 0.006 \left(e^{-0.002y} - e^{-0.003y}\right) \end{aligned}$$

Conditional Distributions

Definition 4

Let $p(x, y)$ be the joint pmf of X and Y.

The conditional pmf of Y given $X = x$ is:

$$\sum_y p_{Y|X}(y) = p_Y(y \mid X = x) = \frac{p(x, y)}{p(x)}, \quad \text{for } x \text{ s.t. } p(x) > 0$$

Note that:

1. $\sum_y p_{Y|x}(y) \geq 0$
2. $\sum_y p_{Y|x}(y) = \sum_y \frac{p(x, y)}{p(x)} = \frac{1}{p(x)} \sum_y p(x, y) = \frac{p(x)}{p(x)} = 1.$
3. $P(Y \in B \mid X = x) = \sum_{y \in \mathbb{B}} P_{Y|X}(y).$

Remark

For continuous random variables, X and Y, we can define

$$p_{Y|x} = \frac{p(x, y)}{p(x)}$$

but $p(x)$ is not a probability, so this definition is not a straightforward application of conditioning on an event. (Please take Measure Theory course; we will slick mostly to the discrete case)

Conditional Expectation and Variance

Definition 5

The **conditional expectation** of Y given $X = x$ is

$$E(Y \mid X = x) = \mu_{Y|x} = \sum_y y P_{Y|X}(y)$$

and the **conditional variance** of Y given X is

$$\text{Var}(Y \mid X = x) = \sigma_Y^2 = \sum_y y^2 P_{Y|x}(y) - \mu_{Y|x}^2$$

Example 6

Cellphone bars:

$$E(Y \mid X = 2) = 4(0.4) + 3(0.4) + 2(0.12) + 1(0.08) = 3.12$$

This is the expected number of repetitions if there are two bars on the cellphone.

$$\begin{aligned}\text{Var}(Y \mid X = 2) &= 4^2(0.4) + 3^2(0.4) + 2^2(0.12) + 1^2(0.08) - (3.12)^2 \\ &= 10.56\end{aligned}$$

Independence

Definition 6

For random variables X and Y, if any one of the following properties is true, then the others are also true, and we say that X and Y are independent:

1. $p(x, y) = p_X(x)p_Y(y)$
2. $p_{Y|x}(y) = p(y)$
3. $p_{X|y}(x) = p(x)$
4. $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A, B

Example 7

Cellphone bars → X, Y are not independent.

Example 8

$Y \backslash X$	1	3	8	$p(y)$
0	0.06	0.1	0.04	0.2
5	0.24	0.4	0.16	0.8
$p(x)$	0.3	0.5	0.2	

Remark
Frequently independence of random variables are assumed from knowledge of the system

Example 9

Two dimensions of a moving mechanical part are independent Gaussian random variables, X and Y with:

$$\begin{aligned} \mu_X &= 10.5 \text{ mm}, & \sigma_X^2 &= 0.0025 \text{ mm} \\ \mu_Y &= 3.2 \text{ mm}, & \sigma_Y^2 &= 0.0036 \text{ mm} \end{aligned}$$

Then:

$$\begin{aligned} &P(10.4 < X < 10.6, 3.15 < Y < 3.25) \\ &\quad = P(10.4 \le X < 10.6)P(3.15 < Y < 3.25) \\ &\quad = P(-2 < z < 2)P(-0.833 < z < 0.833) \\ &\quad = 0.568 \end{aligned}$$

Example 10

Let X, Y be two continuous random variables with joint pdf

$$p(x, y) = 2e^{-x-y}, \quad y \ge x \ge 0$$

The joint pdf factorizes. Are X and Y independent? Let's compute the marginals:

$$\begin{aligned} p(x) &= \int_{-\infty}^{\infty} dy\, p(x, y) \\ &= \int_{x}^{\infty} dy \cdot 2e^{-x-y} = -\,2e^{-x-y}\Big|_{y=x}^{y=\infty} = 2e^{-2x} \qquad , \quad x \ge 0 \end{aligned}$$

$$\begin{aligned} p(y) &= \int_{-\infty}^{\infty} dx\, p(x, y) \\ &= \int_{0}^{y} dx \cdot 2e^{-x-y} = -\,2e^{-x-y}\Big|_{x=0}^{x=y} = 2\left(e^{-y} - e^{-2y}\right) \qquad , \quad y \ge 0 \end{aligned}$$

The conditionals:

$$P_{X|y} = P\frac{(x,y)}{P(y)} = \frac{2e^{-x-y}}{2\left(e^{-y} - e^{-2y}\right)} = \frac{e^{-x}}{1-e^{-y}} \quad , \quad 0 \leq x \leq y$$

$$P_{Y|x} = \frac{P(x,y)}{P(x)} = \frac{2e^{-x-y}}{2e^{-2x}} = e^{x-y} \quad , \quad x \leq y$$

The variables are dependent as a result of the geometrical constraint $p(x,y) \neq 0$ only when $y \geq x \geq 0$.

Example 11: Continued

Let's compute the conditional expectation in the previous example.

$$\begin{aligned} E(X \mid Y = y) = \int_0^y P_{X|y}(x,y) \cdot x\,dx &= \int_0^y \frac{e^{-x}}{1-e^{-y}} x\,dx \\ &= \left.\frac{e^{-x} \cdot x}{1-e^{-y}}\right|_{x=0}^{x=y} + \int_0^y \frac{e^{-x}}{1-e^{-y}} dx \\ &= -\frac{e^{-y} \cdot y}{1-e^{-y}} - \left.\frac{e^{-x}}{1-e^{-y}}\right|_{x=0}^{x=y} \\ &= \frac{1-e^{-y}-ye^{-y}}{1-e^{-y}} \end{aligned}$$

$$\begin{aligned} E\left(Y \mid X = x\right) = \int_x^\infty P_{Y|x}(x,y) y\,dy &= \int_x^\infty e^{x-y} \cdot y\,dy \\ &= -\left. e^{x-y} \cdot y\right|_{y=x}^{y=y} + \int_x^\infty e^{x-y} dy \\ &= x - \left. e^{x-y}\right|_{y=x}^{y=\infty} \\ &= x + 1. \end{aligned}$$