§3. Conditional Probability

Example 1

A card is drawn at random. What is the probability it is a Queen?

Solution

Q = Queen

$$P(Q) = \frac{4}{52} = \frac{1}{13}$$

Example 2

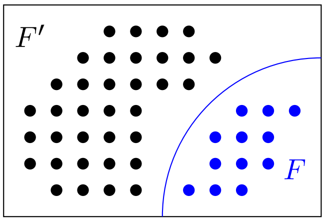
James draws a card and tells us that it is a face card. What is the probability that it is a queen?

Solution

F = Face card
Q = Queen

Knowledge that the event F is realized "contracted" the sample space to F. Observe that

$$P(Q|F) = \frac{P(Q \cap F)}{P(F)} = \frac{4/52}{12/52} = \frac{1}{3}$$



Example 3

Two dice are tossed. What is the probability one die shows 5 if the sum is 7?

Solution

$P = \frac{2}{6} = \frac{1}{3}$ But also,

$$P(OD = 5 \,|\, S = 7) = \frac{P(OD = 5 \cap S = 7)}{P(S = 7)} = \frac{2/36}{6/36} = \frac{1}{3}$$

Example 4

Two dice are tossed. What is the probability the sum is 7 if one die shows 5?

Solution

$$P(S = 7 \,|\, OD = 5) = \frac{PS = 7 \cap OD = 5)}{P(OD = 5)} = \frac{2/36}{11/36} = \frac{2}{11}$$

Definition 1

The **conditional probability** of an event A given that an event B was observed is

$$P(A \,|\, B) = \frac{P(A \cap B)}{P(B)}$$

Note: In general $P(A \,|\, B) \neq P(B \,|\, A)$

Theorem 1: Multiplication Rule

$$P(A \cap B) = P(A \,|\, B)P(B) = P(B \,|\, A)P(A)$$

Example 5

Two cards are drawn at random without replacement. What is the probability that both are spades?

Solution

S_1 = First card is a spade
S_2 = Second card is a spade

$$P(S_1 \cap S_2) = P(S_1)P(S_2 \,|\, S_1) = \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{17}$$

Example 6

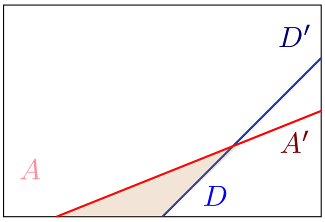
76% of AC flights depart on time (within 5 minutes of schedule). 68% of AC flights arrive on time. 89% of the flights that depart on time arrive on time. Aunt Rose's flight arrived on time. What is the probability it departed on time?

Solution

A = Arrived on time
D = Departed on time

$$P(A\,|\,D)P(D) = P(D\,|\,A)P(A)$$

$$P(D\,|\,A) = \frac{P(A\,|\,D)P(D)}{P(A)} = \frac{(0.89)(0.76)}{0.68} = 0.9947$$



Example 7

What it the probability that Aunt Rose's flight did not depart on time?

Solution

$$P(D'\,|\,A) = 1 - P(D\,|\,A) = 1 - 0.9947 = 0.0053$$

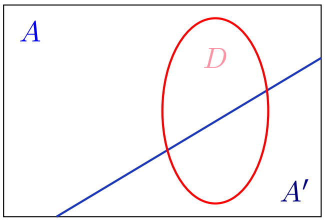
Example 8

Uncle Bob's flight did not arrive on time. What is the probability that it departed on time?

Solution

$$P(D\,|\,A') = \frac{P(D\cap A')}{P(A')} = \frac{P(D) - P(D\cap A)}{1 - P(A)}$$
$$= \frac{(P(D) - P(A\,|\,D)P(D)}{1 - P(A)}$$
$$= \frac{P(D)\,[\,1 - P(A\,|\,D)\,]}{1 - P(A)}$$

$$P(D\,|\,A') = \frac{0.76\,[\,1 - 0.89\,]}{1 - 0.68} = 0.2613$$



$P(B\,|\,A) = \dfrac{P(A\,|\,B)P(B)}{P(A)}$ is sometimes called **Bayes Rule**. It allows to switch the direction of the conditioning.

Example 9

1 in 10 000 people suffers from a rare disease. A test gives positive result for 99% of the people who have the disease and also gives positive result for 5% of the people who do not have the disease.

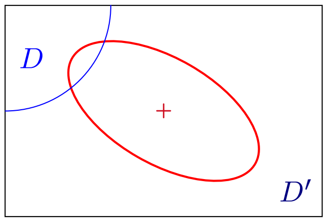
You took the test and tested positive. What is the probability that you have the disease?

Solution

D = Have the disease
$+$ = Positive result

$$P(D) = \frac{1}{10\,000} \qquad P(+\,|\,D) = 0.99 \qquad P(+\,|\,D') = 0.05$$

$$P(D\,|\,+) = \frac{P(+\,|\,D)P(D)}{P(+)} \qquad \text{but } + = (+\cap D)\cup(+\cap D')$$

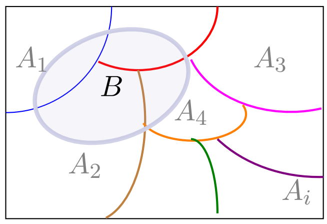


$$\begin{aligned} P(D|+) &= \frac{P(+\,|\,D)P(D)}{P(+\cap D) + P(+\cap D')} \\ &= \frac{P(+\,|\,D)P(D)}{P(+\,|\,D)P(D) + P(+|\,D')P(D')} \\ &= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.05)(0.9999)} = 0.0020 \quad < 0.01 \end{aligned}$$

Theorem 2: Bayes Theorem

If $S = A_1 \cup A_2 \cup \cdots \cup A_n$ with $A_i \cap A_j = \varnothing \quad \forall i \neq j$ (i.e. we have a disjoint partition of the sample space) then

$$P(A_i\,|\,B) = \frac{P(A_i\cap B)}{P(B)} = \frac{P(A_i \cap B)}{\sum_{k=1}^{n} P(A_k \cap B)}$$



$$P(A_i\,|\,B) = \frac{P(B\,|\,A_i)P(A_i)}{\sum_{k=1}^{n} P(B\,|\,A_k)P(A_k)}$$

Example 10

The CFA exam has three levels. At an exam location 60% of the examinees write L_1; 30% write L_2; and 10% write L_3. The failure rates are

$$P(F|L_1) = 0.59, \qquad P(F|L_2) = 0.56, \qquad P(F|L_3) = 0.44$$

An examinee leaves the exam location dejected; certain that they failed. What are the probabilities this examinee to L_1? L_2? L_3?

Solution

$$P(L_1 \mid F) = \frac{P(F \mid L_1)P(L_1)}{P(F|L_1)P(L_1) + P(F|L_2)P(L_2) + P(F|L_3)P(L_3)}$$

$$= \frac{(0.59)(0.6)}{(0.59)(0.6) + (0.56)(0.3) + (0.44)(0.1)}$$

$$= \frac{(0.354)(0.3)}{0.566} = 0.625$$

$$P(L_2 \mid F) = \frac{P(F \mid L_2)P(L_2)}{P(F \mid L_1)P(L_1) + P(F \mid L_2)P(L_2) + P(F \mid L_3)P(L_3)}$$

$$= \frac{(0.56)(0.3)}{0.566} = 0.297$$

$$P(L_3 \mid F) = \frac{P(F \mid L_3)P(L_3)}{P(F \mid L_1)P(L_1) + P(F \mid L_2)P(L_2) + P(F \mid L_3)P(L_3)}$$

$$= \frac{(0.44)(0.1)}{0.566} = 0.078$$

Example 11: Monty Hall

A game show contestant is shown three doors. Behind one of the doors is a car and the other two doors hide goats. The contestant chooses a door. The host opens a door showing a goat. Should the contestant keep their original choice or switch?

Solution

With the original choice:

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$$

i.e. the car is equally likely to be behind any of the three doors.

Suppose that the contestant chose Door #1 and the host opens Door #2. Let D_2 = Host opened Door #2.

$$P(D_2|C_1) = \frac{1}{2} \qquad P(D_2|C_2) = 0 \qquad P(D_2 \,|\, C_3) = 1$$

$$\begin{aligned} P(C_3 \,|\, D_2) &= \frac{P(D_2 \,|\, C_3)P(C_3)}{P(D_2 \,|\, C_1)P(C_1) + P(D_2 \,|\, C_2)P(C_2) + P(D_3 \,|\, C_3)P(C_3)} \\ &= \frac{1(1/3)}{(1/2)(1/3) + 0(1/3) + 1(1/3)} \\ &= \frac{2}{3} \end{aligned}$$

The contestant should switch.