§4. Independence

Definition 1

Two events, A and B, are **independent** if

$$P(A \cap B) = P(A)P(B)$$

Equivalently

$$P(A \mid B) = P(A) \qquad \text{or} \qquad P(B \mid A) = P(B)$$

Example 1

Draw a card, observe, and put it back. Reshuffle, draw another card, and observe. What is the probability that we observe two aces?

Solution

A_1 = First card is an ace.
A_2 = Second card is an ace.

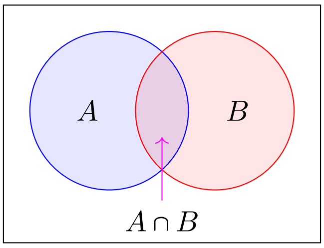
$$P(A_1 \cap A_2) = P(A_1)P(A_2) = \frac{4}{52} \cdot \frac{4}{52}$$

Remark
Without replacement

$$P(A_1 \cap A_2) = P(A_2 \mid A_1)P(A_2) = \frac{3}{51} \cdot \frac{4}{52}$$

Note: Disjoint events $A \cap B = \varnothing$ are not independent at all:

$$0 = P(\varnothing) = P(A \cap B) \neq P(A)P(B)$$



$$P(A) = P(A \mid B)$$

Proposition 1

If A and B are independent, then A' and B' are independent, and so are A, B' and A', B.

> **Remark**
> $P(A) = P(A|B)$, $P(B) = P(B|A)$, neither A is informative of B, nor is B informative of A

Proof.

$$\begin{aligned} P(A' \cap B') = P(A \cup B)' &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= [1 - P(A)][1 - P(B)] \\ &= P(A')P(B') \end{aligned}$$

□

Definition 2

n–events A_i are independent if

$$P(B_1 \cap B_2 \cap \cdots \cap B_n) = P(B_1)P(B_2) \cdots P(B_n),$$

where each B_i is either A_i or A_i'. Thus n events are independent if the joint probability for any combination of the events and their complements factorizes.

> **Remark**
> Pairwise independence is not sufficient for independence.

Example 2

Toss four coins and consider the events:

A = First coin shows heads
B = Third coin shows tails
C = There are an equal number of heads and tails

Then $|S| = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{6}{16} = \frac{3}{8}$$

$$P(A \cap B) = \frac{4}{16} = \frac{1}{4} = P(A)P(B)$$

$$P(A \cap C) = \frac{3}{16} = P(A)P(C)$$

$$P(B \cap C) = \frac{3}{16} = P(B)P(C)$$

The three events are pairwise independent. However,

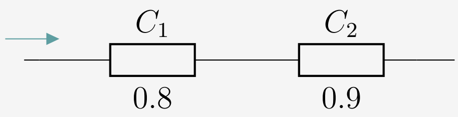
$$P(A \cap B \cap C) = \frac{2}{16} \neq P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{8}$$

$\Rightarrow$ so the three events are not independent.

In the analysis of reliability of systems, components are assumed to fail independently.

Example 3: Series Circuit

Suppose that the probability of that Component #1 is functioning properly is 0.8 and Component #2 is functioning properly is 0.9.



What is the probability that the system is functioning?

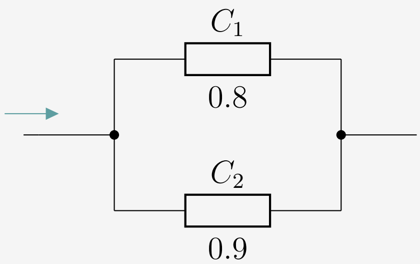
Solution

$P(C_1)$ = Component #1 is functioning
$P(C_2)$ = Component #2 is functioning
$P(\text{sys})$ = System is functioning

$$P(\text{sys}) = P(C_1 \cap C_2) = P(C_1)P(C_2) = (0.8)(0.9) = 0.72$$

Example 4: Parallel Circuit

Suppose that the probability of that Component #1 is functioning properly is 0.8 and Component #2 is functioning properly is 0.9.



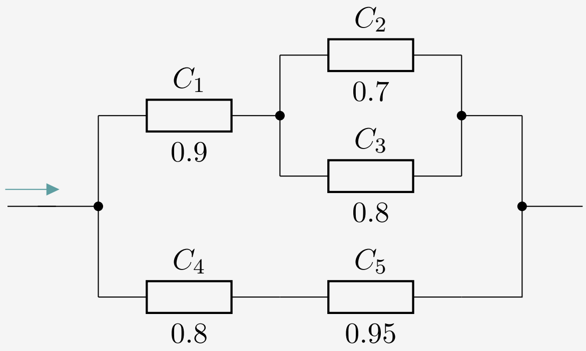
What is the probability that the system is functioning?

Solution

$$\begin{aligned} P(\text{sys}) = P(C_1 \cup C_2) &= P(C_1) + P(C_2) - P(C_1 \cap C_2) \\ &= P(C_1) + P(C_2) - P(C_1)P(C_2) \\ &= 0.8 + 0.9 - 0.72 \\ &= 0.98 \end{aligned}$$

Example 5: More Complicated System

Consider the following system. What is the probability that the system is functioning?



Solution

$$\begin{aligned}
P(\text{sys}) &= P(\,[C_1 \cap (C_2 \cup C_3)\,] \cup [\,(C_4 \cap C_5)\,]) \\
&= P[\,(C_1 \cap C_2) \cup (C_1 \cap C_3) \cup (C_4 \cap C_5)\,] \\
&= P(C_1 \cap C_2) + P(C_1 \cap C_3) + P(C_4 \cap C_5) - P(C_1 \cap C_2 \cap C_3) \\
&\quad - P(C_1 \cap C_2 \cap C_4 \cap C_5) - P(C_1 \cap C_3 \cap C_4 \cap C_5) \\
&\qquad + P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5) \\
&= (0.9)(0.7) + (0.9)(0.8) + (0.8)(0.95) - (0.9)(0.7)(0.8) \\
&\quad - (0.9)(0.7)(0.8)(0.95) - (0.9)(0.8)(0.8)(0.95) \\
&\qquad + (0.9)(0.7)(0.8)(0.8)(0.95) \\
&= 0.96304
\end{aligned}$$

Example 6

The system in the previous problem is functioning properly. What is the probability that the first component has failed?

Solution

$$\begin{aligned}
P(C_1' \,|\, \text{sys}) &= \frac{P(C_1' \cap \text{sys})}{P(\text{sys})} \\
&= \frac{1}{P(\text{sys})}\left[\cancelto{0}{P(C_1' \cap C_1 \cap C_2)} + \cancelto{0}{P(C_1' \cap C_1 \cap C_3)} + P(C_1' \cap C_4 \cap C_5)\right] \\
&= \frac{1}{P(\text{sys})} P(C_1') P(C_4) P(C_5) \\
&= \frac{(0.1)(0.8)(0.95)}{0.96304} = 0.07892
\end{aligned}$$