§5. Counting

Principle of Indifference

Recall:

$$P(A) = \frac{|A|}{|S|}$$

Principle 1:

If A and B are independent events and A can occur in m ways and B can occur in n ways, then A **and** B together can occur in $m \times n$ ways.

Principle 2:

If A and B are disjoint events then A **or** B can occur in $m + n$ ways.

Example 1

How many old style Quebec license plates? New style? Electric vehicle license plates?

Solution

Old:

$$10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 17\,576\,000$$

New:

$$26 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 45\,697\,600$$

EV:

$$26 \cdot 10 \cdot 10 \cdot 1 \cdot 26 \cdot 26 = 1\,757\,600$$

Old License Plate

N N N L L L

New License Plate

L N N L L L

EV License Plate

L N N V L L

Permutations

Example 2

How many words can be formed with the letters A, B, and C with no repetition?

Solution

$$3 \cdot 2 \cdot 1 = 6 = 3!$$

ABC, ACB, BAC, BCA, CAB, CBA

Example 3

How many words with n distinct letters? $\longrightarrow$ $n!$

The number of words formed using 6 different letters with no repetitions is $6! = 720$ and for 66 distinct letters it would be $66! = 5.44 \times 10^{92}$

Example 4

How many words of length 2 can be formed using the letter A, B, C, D, and E with no repetitions allowed?

Solution

$$5 \cdot 4 = 20 = \frac{5!}{(5-2)!} = \frac{5!}{3!}$$

AB, AC, AD, AE, BA, BC, BD,...

Definition 1

The number of **permutations** of r objects selected from n distinct objects is

$$_nP_r = P_r^n = \frac{n!}{(n-r)!}$$

In particular, when we are selecting all n objects we have

$$_nP_n = P_n^n = \frac{n!}{0!} = n!$$

Example 5

32 finalists are vie for the FIFA World Cup. In how many ways could the medals be given?

$$_{32}P_3 = \frac{32}{(32-3)!} = 29\,760$$

Example 6

The letters A, B, C, D, E, F, G, H are randomly permuted. What is the probability that the resulting word ends with A?

Solution

A = the word ends with A

$$P(A) = \frac{7!}{8!} = \frac{1}{8}$$

_ _ _ _ _ _ _ A

Permuting Non-distinct Objects

Example 7

How many permutations of A,A,B,C?

$$\frac{4!}{2!} = 12$$

AABC, ABCA, BCAA, CAAB, AACB, ACBA, CBAA, BAAC, BACA, ACAB, ABAC, CABA

How many permutations of A,A,B,B?

$$\frac{4!}{2!2!} = 6$$

AABB, ABAB, ABBA, BBAA, BABA, BAAB

In general:

If a set of n objects has n_1 objects of Type 1, n_2 objects of Type 2, ..., n_k objects of Type k, then there are

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

distinct permutations of the set of n objects.

Example 8

In how many ways can a group of 16 programmers be assigned to three jobs requiring 7, 5, and 4 programmers respectively?

$$\frac{16!}{7!5!4!} = 1\,441\,440$$

Example 9

In how many ways can a group of 6 numbers be selected from 49 in the biweekly lotto drawing? What is the probability of winning the Lotto 649 on one selection?

Solution

Number of ways that 6 numbers can be selected from 49:

$$\frac{49!}{6!43!} = 13\,983\,816$$

Probability of winning with one selection:

$$P(\text{win}) = \frac{1}{13\,983\,816}$$

Combinations

Definition 2

The number of selections of r objects out of n without regard for order of the selection (selection as a group) is

$$_nC_r = C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 10

A engineering firm has 12 female engineers and 15 male engineers. 9 engineers are selected at random and will work on a project in Patagonia. What is the probability the team selected has 6 females and 3 males on it?

$$P(\,6\,\text{females and }3\,\text{males}\,) = \frac{_{12}C_6 \cdot {_{15}C_3}}{_{27}C_9} = \frac{924 \cdot 455}{4\,686\,825} = 0.0897$$

Example 11

$$
\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}
$$

$$
\begin{aligned}
(x+y)^4 &= x^4 +_4 C_1 \cdot x^3y +_4 C_2 \cdot x^2y^2 +_4 C_3 \cdot xy^3 + y^4 \\
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\end{aligned}
$$

Theorem 1: The Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} {}_nC_r\, x^r y^{n-r}$$

Example 12

$$(1+1)^n = 2^n = \sum_{r=0}^{n} {}_nC_r$$

Example 13

Five red and four blue marbles are arranged in a row. What is the probability that both end marbles are blue?

Solution

B = Both end marbles are blue

$$P(B) = \frac{{}_7C_5}{{}_9C_5} = \frac{21}{126} = \frac{1}{6}$$

Example 14

Ten race cars, numbered 1 to 10, are racing around a track. A spectator observes three cars in a row. If the cars appear in random order, what is the probability that the largest number seen on a car is 6?

Solution

$C6$ = Largest number is a six

$$P(C6) = \frac{1\,{}_5C_2}{{}_{10}C_3} = \frac{10}{120} = \frac{1}{12}$$

Example 15

A deck of cards is well shuffled. The cards are dealt one at a time until the first ace appears.

a. Find the probability that no King, Queen, or Jack appears before the first ace.

b. Find the probability that exactly one King, one Queen, and one Jack appears (in any order) before the first ace.

Solution

a. N = No face card appears before the first ace

$$P(N) = \frac{1}{4}$$

$\because$ an ace, king, queen, or jack are equally likely.

b. E = Exactly one king, one queen, and one jack appears before the first ace

$$P(E) = \frac{3! \cdot 4^4 \cdot 12!}{16!} = 0.0352$$

Example 16

What is the probability that a poker hand is a "three of a kind"?

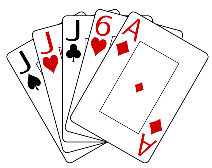

Figure 1: Three of a kind

Solution

T = Three of a kind

$$P(T) = \frac{1}{{}_{52}C_5}\left[{}_{13}C_1 \cdot {}_4C_3 \cdot {}_{12}C_2 \cdot ({}_4C_1)^2\right] = 0.0211$$

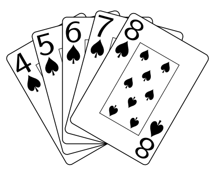

Figure 2: Straight Flush

Example 17

What is the probability that a poker hand is a "straight flush" but not a "Royal flush"?

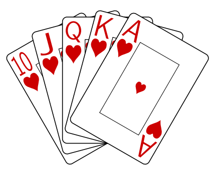

Figure 3: Royal Flush

Solution

S = Straight flush

$$P(S) = \frac{1}{{}_{52}C_5}\left[{}_{10}C_1 \cdot {}_4C_1 - 1\,{}_4C_1\right] = 1.39 \times 10^{-5}$$