§6. Discrete Random Variables

In many situations the sample points of a random experiment are labelled with numerical values (possibly vectorial).

Example 1

Toss a die: $S = \{1, 2, 3, 4, 5, 6\}$

Toss two dice: $S\left\{\binom{1}{1}, \binom{1}{2}, \binom{1}{3}, \cdots \quad \binom{6}{5}, \binom{6}{6}\right\}$

Non-example: Toss a coin $S = \{H, T\}$, but we can relabel to $S = \{0, 1\}$

Remark
When the outcomes of a random experiment have associated nuumerical values the power of arithmetic, calculus, analysis, become available for modelling, inference, prediction, ...

Definition 1

A **random variable** (RV), X, is a function from the sample space S into the real numbers:

$$X : S \to \mathbb{R}$$

- a multivariate (vectorial) random variable is $X : S \to \mathbb{R}^n$
- a discrete RV is $X : S \to \mathbb{N}$ or $X : S \to \mathbb{Z}$ or $\ldots$

Example 2

The number of attempts until an exam is passed is a discrete RV.

$$P \to 1; \quad FP \to 2; \quad FFP \to 3; \quad FFFP \to 4; \ldots$$

Notice that S is infinite. Say at any attempt that $P(P) = P(F) = \frac{1}{2}$

X	1	2	3	4	5	$\ldots$
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\ldots$

Definition 2

Let $X : S \to \mathbb{Z}$ be a discrete RV. The **probability mass function** (pmf) of X is

$$p(x) = P(X = x)$$

Remark
Notice that

1. $p(x) \geq 0$
2. $\sum_S p(x) = 1$

Example 3

Toss a die. The pmf is

X	1	2	3	4	5	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Remark This is an example of a uniform distribution

Example 4

Toss tow dice and observe the sum.

X	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Example 5

Pull four cards at random without replacement.

Let X = the number of diamonds.

X	0	1	2	3	4
$p(x)$	$\frac{{}_{13}C_0\,{}_{39}C_4}{{}_{52}C_4} = 0.3038$	$\frac{{}_{13}C_1\,{}_{39}C_3}{{}_{52}C_4} = 0.4388$	$\frac{{}_{13}C_2\,{}_{39}C_2}{{}_{52}C_4} = 0.2135$	$\frac{{}_{13}C_3\,{}_{39}C_1}{{}_{52}C_4} = 0.0412$	$\frac{{}_{13}C_4\,{}_{39}C_0}{{}_{52}C_4} = 0.0027$

Definition 3

Let X be a discrete random variable. Its **cumulative distribution function** (also known as a distribution function)is

$$F(x) = P(X \le x) = \sum_{t \le x} p(t)$$

Example 6

Pull four cards at random without replacement.

Let X = the number of diamonds. The cumulative distribution is

X	0	1	2	3	4
$F(x)$	0.3038	0.7426	0.9561	0.9973	1.0000