§7. Expected Value and Variance of Discrete Random Variables

Definition 1

Let X be a discrete random variable with pmf, $p(x)$. The **expected value** (mean) of X is defined as

$$E(X) = \mu_x = \sum_x x \cdot p(x)$$

Remark
The sum may not converge.

Example 1

Toss a die. Then

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

Example 2

Toss two dice. What is the expected value of the sum?

$$\begin{aligned} E(X) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} \\ &\qquad + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} \\ &= \frac{28}{4} \\ &= 7 \end{aligned}$$

$$\begin{aligned} E(X_1) + E(X_2) &= 2(3.5) \\ &= 7 \end{aligned}$$

Example 3

Pull four cards at random without replacement.

Let X = the number of spades.

X	0	1	2	3	4
$p(x)$	0.3038	0.4388	0.2135	0.0412	0.0027

Expected number of spades:

$$\begin{aligned} E(X) &= 0\cdot(0.3038) + 1\cdot(0.4388) + 2\cdot(0.2135) + 3\cdot(0.0412) \\ &\quad + 4\cdot(0.0027) \\ &= 1 \end{aligned}$$

Example 4

Suppose that the probability of a passing an exam is p.

Let X = the number of attempts until a pass.

X	1	2	3	4	$\cdots$	k	$\cdots$
$p(x)$	p	$(1-p)\cdot p$	$(1-p)^2\cdot p$	$(1-p)^3\cdot p$	$\cdots$	$(1-p)^k\cdot p$	$\cdots$

Expected number of attempts until a pass is then:

$$\begin{aligned} E(X) &= \sum_{x=1}^{\infty} x\cdot(1-p)^{x-1}\cdot p \\ &= 1\cdot p + 2\cdot(1-p)\cdot p + 3\cdot(1-p)^2\cdot p + \cdots \end{aligned}$$

$$(1-p)E(X) = 1\cdot(1-p)\cdot p + 2\cdot(1-p)^2\cdot p + 3\cdot(1-p)^3\cdot p + \cdots$$

$$\begin{aligned} E(X) - (1-p)E(X) &= 1\cdot p + 1\cdot(1-p)\cdot p + 1\cdot(1-p)^2\cdot p + \cdots \\ &= \sum_{x=1}^{\infty}(1-p)^{x-1}\cdot p \\ &= \sum_{x=1}^{\infty} p(x) \\ &= 1 \end{aligned}$$

$$\begin{aligned} E(X)[1-1+p] &= 1 \\ E(X) &= \frac{1}{p} \end{aligned}$$

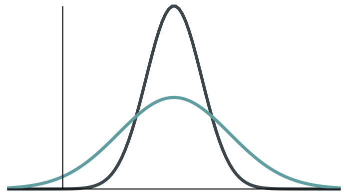
If $p = 0.2$ then $E(X) = \frac{1}{0.2} = 5$

Variance of a Discrete RV

If we consider $E(x-\mu)$ we get

$$E(X-\mu) = E(X) - \underbrace{E(\mu)}_{\mu\sum_x p(x)} = \mu - \mu = 0$$

$E|X-\mu|$ is better, but uncommon.



Scatter around $E(X)$

Definition 2

The **variance** of the random variable X is defined as

$$Var(X) = \sigma_x^2 = E(x-\mu)^2 = \sum_x (x-\mu)^2 \cdot p(x)$$

The **standard deviation** of X is

$$\sigma_x = \sqrt{Var(X)}$$

Example 5

Toss a die.

$$\begin{aligned}\sigma_x^2 &= \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \frac{1}{6}(3-3.5)^2 + \frac{1}{6}(4-3.5)^2 + \frac{1}{6}(5-3.5)^2 \\ &\qquad + \frac{1}{6}(6-3.5)^2 \\ &= 2.917\end{aligned}$$

$$\sigma_x = \sqrt{2.917} = 1.708$$

Remark

$$\begin{aligned}\sigma_x^2 &= E(X-\mu)^2 \\ &= E(X^2 - 2\mu X - \mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu\mu + \mu^2 \\ &= E(X^2) - E(X)^2\end{aligned}$$

Example 6

Redo: toss of a die.

$$\begin{aligned}\sigma_x^2 &= E(X^2) - \mu_x^2 \\ &= 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{6} + 3^2\cdot\frac{1}{6} + 4^2\cdot\frac{1}{6} + 5^2\cdot\frac{1}{6} + 6^2\cdot\frac{1}{6} - \left(\frac{7}{2}\right)^2 \\ &= \frac{91}{6} - \frac{49}{4} \\ &= \frac{35}{12} \\ &= 2.917\end{aligned}$$

Example 7

Pull four cards at random without replacement.

Let X = the number of spades.

X	0	1	2	3	4
$p(x)$	0.3038	0.4388	0.2135	0.0412	0.0027

$\mu_x = 1$

Then

$$\begin{aligned} Var(X) &= 0^2(0.3038) + 1^2(0.4388) + \cdots + 4^2(0.0027) - 1^2 \\ &= 0.7068 \end{aligned}$$

$$\begin{aligned} \sigma_x &= \sqrt{0.7068} \\ &= 0.8407 \end{aligned}$$

Example 8

Suppose that the probability of passing an exam is p.
Let X = number of attempts until passing an exam.
Then $p(x) = (1-p)^{x-1} \cdot p$; $E(x) = \frac{1}{p}$.

$$\begin{aligned} E(X(X-1)) &= E(X^2) - E(X) \\ &= 1 \cdot 0 \cdot p + 2 \cdot 1 \cdot (1-p) \cdot p + 3 \cdot 2 \cdot (1-p)^2 \cdot p + \cdots \end{aligned}$$

$$(1-p)E(X(X-1)) = 2 \cdot 1 \cdot (1-p)^2 \cdot p + 3 \cdot 2 \cdot (1-p)^3 \cdot p + \cdots$$

$$\begin{aligned} pE(X(X-1)) &= E(X(X-1)) - (1-p)E(X(X-1)) \\ &= 2(1-p)p + 4(1-p)^2p + 6(1-p)^3p + \cdots \\ &= 2(1-p)\left[p + 2(1-p)p + 3(1-p)^2p + \cdots\right] \\ &= 2(1-p)E(X) \end{aligned}$$

$$\begin{aligned} E(X(X-1)) &= E(X^2) - E(X) \\ &= \frac{2(1-p)}{p}E(X) \\ E(X^2) &= E(X)\left[\frac{2(1-p)}{p} + 1\right] \\ &= E(X) \cdot \frac{2+p}{p} \end{aligned}$$

$$\sigma_X^2 = E(X^2) - E(X)^2$$
$$= \frac{1}{p} \cdot \frac{2-p}{p} - \frac{1}{p^2} = \frac{1-p}{p^2} \qquad \Rightarrow \qquad \sigma_x = \frac{\sqrt{1-p}}{p}$$

Chebyshev Inequality

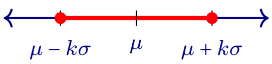
How does σ_x measure variability?

Theorem 1

Suppose that the random variable X has mean, μ, and standard deviation, σ. then

$$P(|X-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2} \qquad \forall k > 0$$

Remark
The interval, contains at least $1 - \frac{1}{k^2}$ of the probability.



For $k = 2$:

$$1 - \frac{1}{k^2} = 0.75$$

probability of being 2σ from μ

Proof. Let $A = \{x \mid \ |X-\mu| > k\sigma\}$ and $B = \{x \mid \ |X-\mu| \le k\sigma\}$

$$\sigma^2 = \sum_{x \in A} (X-\mu)^2 \cdot p(x) + \sum_{x \in B} (X-\mu)^2 \cdot p(x) \ge \sum_{x \in A} (k\sigma)^2 \cdot p(x) + \sum_{x \in B} 0 \cdot p(x)$$

Note: $P(A) = P(|X-\mu| > k \cdot \sigma)$

$$\cancel{\sigma^2} \ge k^2 \cdot \cancel{\sigma^2} P(|X-\mu| > k\sigma)$$
$$\frac{1}{k^2} \ge P(|X-\mu| > k\sigma)$$
$$1 - \frac{1}{k^2} \le 1 - P(|X-\mu| > k\sigma) = P(|X-\mu| \le k\sigma)$$

□