§8.Binomial Random Variables

Bernoulli trials: a sequence of random experiments such that:

- only two outcomes occur: success or failure
- the trials are independent
- $p = P(S)$ is constant throughout the trials; $P(F) = 1 - p$ is also constant.

Example 1

10% of all people are lefties.

a. What is the probability that two people in a random sample of $n = 7$ will be lefties?

b. What is the probability that x people in a a random sample of $n = 7$ will be lefties?

Solution

a. $P(2) = {}_7C_2\,(0.1)^2(0.9)^5 = 0.1240$

b. $P(x) = {}_7C_x\,(0.1)^x(0.9)^{7-x}$

X	0	1	2	3	4	5	6	7
$p(x)$	0.4783	0.3720	0.1240	0.0230	0.0026	0.0002	6.3×10^{-6}	1×10^{-7}

Definition 1

A **binomial random variable** with parameters, n, (number of trials) and p, (probability of success) is a discrete random variable with pmf

$$P(x) = {}_nC_x\,p^x(1-p)^{n-x} \qquad , \qquad x = 0, 1, 2, \ldots, n$$

Note:

$$\sum_{x=1}^{n} p(x) = \sum_{x=1}^{n} {}_nC_x\,p^x(1-p)^{n-x} = (p + (1-p))^n = 1^n = 1$$

Example 2

When given a choice between chicken or beef, 75% of people choose chicken. What is the probability that on a flight with 60 passengers, less than 40 will choose chicken?

Solution

X = number of people who choose chicken.

$$P(X < 40) = \sum_{x=0}^{39} {}_{60}C_x (0.75)^x (0.25)^{60-x} = 0.0541$$

Example 3

What is the probability that more than 20 will choose beef?

$\rightarrow p = 0.0541$

Expected Value and Variance of $X \sim$ Binom(n, p)

Theorem 1

If $X \sim \text{Binom}(n, p)$ then;

$$E(X) = np \qquad Var(X) = np(1-p)$$

Proof. Let's prove the result for the mean:

$$\begin{aligned}
E(X) = \sum_{x=0}^{n} x \cdot p(x) = \sum_{x=1}^{n} x \cdot p(x) &= \sum_{x=1}^{n} {}_nC_x \, x p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n!}{(n-x)!x!} x p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} n \cdot \frac{(n-1)!}{(n-x)!(x-1)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!(x-1)!} p^{x-1} (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} {}_{n-1}C_{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)} \\
&= np \cdot 1 \\
&= np
\end{aligned}$$

□

Example 4

75% of people prefer chicken and 25% prefer beef. On a flight with 60 people what is the expected value and the standard deviation of people who prefer chicken?

Solution

X = number of people who prefer chicken.

$$E(X) = np = 60(0.75) = 45$$

$$Var(X) = np(1-p) = 60(0.75)(0.25) = 11.25 \quad ; \quad \sigma_x = \sqrt{11.25} = 3.354$$